Lecture 5.2: Boundary conditions for the heat equation

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Last time: Example 1a

The solution to the following B/IVP for the heat equation:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = x(1-x)\,.$$

is $u(x,t) = \displaystyle\sum_{n=1}^{\infty} \tfrac{4(1-(-1)^n)}{n^3\pi^3} \sin(n\pi x)\, e^{-(cn\pi)^2 t}$.

This time: Example 1b

Solve the following B/IVP for the heat equation:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = u(1,t) = 32, \qquad u(x,0) = x(1-x) + 32\,.$$

Last time: Example 1a

The solution to the following B/IVP for the heat equation:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = x(1-x)\,.$$

is $u(x,t) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^3\pi^3} \sin(n\pi x)\, e^{-(cn\pi)^2 t}$.

This time: Example 1c

Solve the following B/IVP for the heat equation:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = 32, \quad u(1,t) = 42, \qquad u(x,0) = x(1-x) + 32 + 10x\,.$$

A familiar theme

Summary

To solve the initial / boundary value problem

$$u_t = c^2 u_{xx}, \qquad u(0,t) = a, \quad u(L,t) = b, \qquad u(x,0) = h(x)\,,$$

first solve the related homogeneous problem, then add this to the steady-state solution $u_{ss}(x) = a + \frac{b-a}{L}x$.

Neumann boundary conditions (type 2)

Example 2

Solve the following B/IVP for the heat equation:

$$u_t = c^2 u_{xx}, \qquad u_x(0,t) = u_x(1,t) = 0, \qquad u(x,0) = x(1-x)\,.$$

Neumann boundary conditions (type 2)

Example 2 (cont.)

The general solution to the following BVP for the heat equation:

$$u_t = c^2 u_{xx}, \qquad u_x(0,t) = u_x(1,t) = 0, \qquad u(x,0) = x(1-x)\,.$$

is $u(x,t) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos(n\pi x)\, e^{-(cn\pi)^2 t}$. Now, we'll solve the remaining IVP.

Mixed boundary conditions

Example 1.5

Solve the following B/IVP for the heat equation:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = u_x(1,t) = 0, \qquad u(x,0) = 5\sin(\pi x/2)\,.$$

Periodic boundary conditions

Example

Solve the following B/IVP for the heat equation:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = u(2\pi,t), \qquad u(x,0) = 2 + \cos x - 3\sin 2x\,.$$