Lecture 6.4: Solving PDEs with Fourier transforms

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4340, Advanced Engineering Mathematics

Introduction

Let $\mathcal{S}$ be subspace of $\mathcal{C}^\infty(\mathbb{R})$ consisting of functions that decay as $x \to \pm\infty$ faster than any polynomial:

$$\mathcal{S} = \left\{ u \in \mathcal{C}^\infty(\mathbb{R}) : \left| \frac{d^k u}{dx^k} \right| \leq C|x|^{-n} \text{ as } |x| \to \infty, \qquad \forall k \in \mathbb{N},\ n \in \mathbb{Z} \right\}.$$

These are the Schwartz class of functions, and $u \in \mathcal{S}$ iff $\widehat{u} \in \mathcal{S}$.

Definition

The Fourier transform of $f \in \mathcal{S}$ is defined by

$$\mathcal{F}f = \widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx.$$

Fourier transforms and its properties

Definition

For a function $u(x,t)$ of two variables, define its Fourier transform by

$$(\mathcal{F}u)(\omega,t)=\widehat{f}(\omega,t)=\int_{-\infty}^{\infty}u(x,t)e^{-i\omega x}\,dx.$$

Remark

The Fourier transform turns x-derivatives into multiplication, and leaves t-derivatives unchanged:

- $(\mathcal{F}u_x)(\omega,t)=(-i\omega)\widehat{u}(\omega,t)$
- $(\mathcal{F}u_{xx})(\omega,t)=(-i\omega)^2\widehat{u}(\omega,t)$
- $(\mathcal{F}u_t)(\omega,t)=\widehat{u}_t(\omega,t)$.

Convolution theorem

For functions f and g,

$$\mathcal{F}(f*g)(\omega)=\widehat{f}(\omega)\widehat{g}(\omega).$$

By taking the inverse Fourier transform of both sides, it follows that

$$(f*g)(x)=\mathcal{F}^{-1}\big(\widehat{f}(\omega)\widehat{g}(\omega)\big).$$

Fourier transform of the Gaussian function

Example 1

Compute the Fourier transform for the Gaussian function $u(x) = e^{-ax^2}$, where $a > 0$.

Solving an ODE with the Fourier transform

Example 2

Solve the following ODE for $u(x)$, given some forcing term $f(x) \in \mathcal{S}$:

$$u'' - u = f(x).$$

Solving a PDE with the Fourier transform

Example 3

Solve the following Cauchy problem for the heat equation, given some $f(x) \in \mathcal{S}$:

$$u_t - ku_{xx} = 0, \qquad u(x,0) = f(x).$$