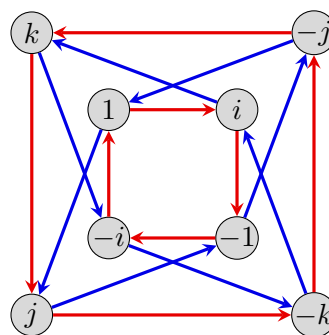
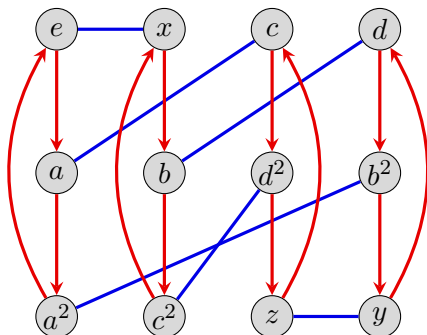


Read the following, which can all be found either in the textbook or on the course website.

- Chapters 5 of *Visual Group Theory* (VGT).
- VGT Exercises 5.3, 5.5, 5.6–5.9, 5.12, 5.14, 5.16, 5.21, 5.25–5.27, 5.29, 5.36, 5.42, 5.44.

Write up solutions to the following exercises.

1. Carry out the following steps for the groups A_4 and Q_8 , whose Cayley graphs are shown below.



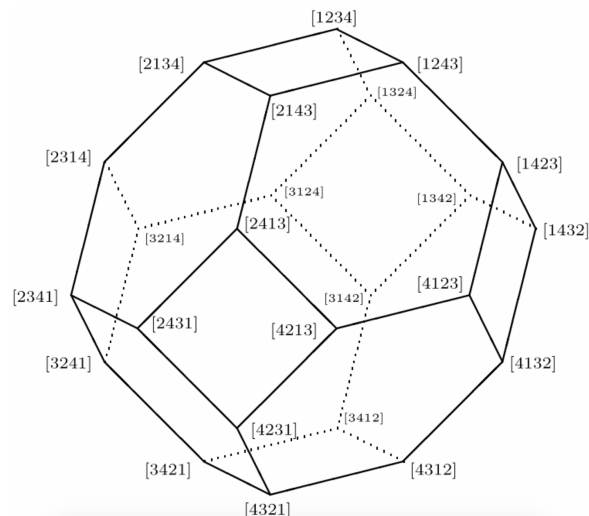
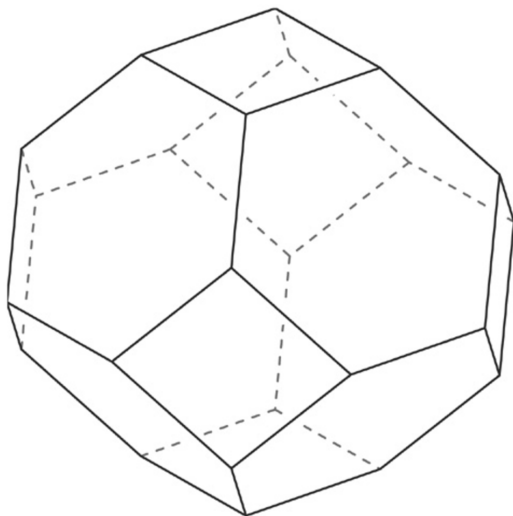
- (a) Find the orbit of each element.
 - (b) Draw the orbit graph of the group.
2. Prove algebraically that if $g^2 = e$ for every element of a group G , then G must be abelian.
 3. Compute the product of the following permutations. Your answer for each should be a single permutation written in cycle notation as a product of disjoint cycles.
 - (a) $(1\ 3\ 2)(1\ 2\ 5\ 4)(1\ 5\ 3)$ in S_5 ;
 - (b) $(1\ 5)(1\ 2\ 4\ 6)(1\ 5\ 4\ 2\ 6\ 3)$ in S_6 .
 4. Write out all $4! = 24$ permutations in S_4 in cycle notation as a product of disjoint cycles. Additionally, write each as a product of transpositions, and decide if they are even or odd. Which of these permutations are also in A_4 ?
 5. (a) The group S_3 can be generated by the transpositions $(1\ 2)$ and $(2\ 3)$. In fact, it has the following presentation

$$S_3 = \langle a, b \mid a^2 = e, b^2 = e, (ab)^3 = e \rangle,$$

where one can take $a = (1\ 2)$ and $b = (2\ 3)$. Make a Cayley diagram for S_3 using this generating set.

- (b) Make a Cayley diagram for the group generated by the permutations $a = (1\ 2)$ and $c = (3\ 4)$, and write down a group presentation for this.

- (c) The group S_4 can be generated by the transpositions $(1\ 2)$, $(2\ 3)$, and $(3\ 4)$. Make a Cayley diagram for S_4 using this generating set. This can be laid out on a polytope called a *permutahedron*, which is a truncated octahedron shown below. On the right, the vertices are labeled with the corresponding permutations in one-line notation. Make a Cayley graph by labeling the vertices on the unlabeled permutahedron with the 24 permutations of S_4 in cycle notation, and color the edges appropriately.



- (d) Write down a group presentation for S_4 using the generating set in Part (b).
6. The Cayley diagram for A_4 shown above labels the elements with letters instead of permutations:

$$A_4 = \{e, a, a^2, b, b^2, c, c^2, d, d^2, x, y, z\}.$$

Redraw this Cayley diagram but label the nodes with the 12 even permutations from the previous problem. That is, you need to determine which permutation corresponds to a , which to b , and so on. [*Hint*: There are many possible ways to do this. If you let a be one of the permutations of order 3, and let x be one of the permutations of order 2, then you should be able to determine the remaining elements.]