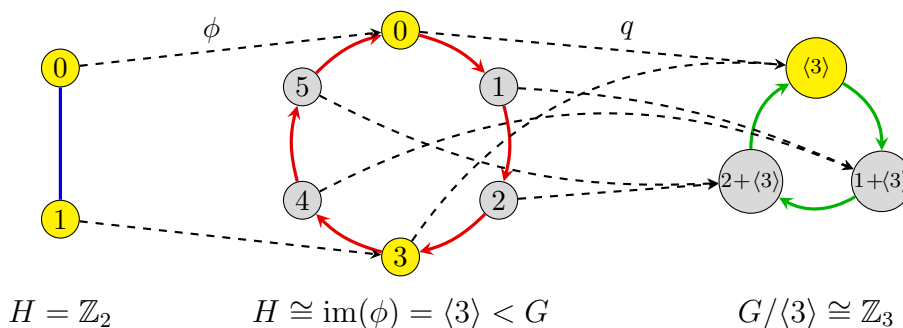


Read the following, which can all be found either in the textbook or on the course website.

- Chapters 8.4–5 of *Visual Group Theory*, or Chapter 9.2 of *IBL Abstract Algebra*.
- VGT Exercises 8.19, 8.23, 8/26, 8.27, 8.37(ab), 8.41, 8.43–8.50.

Write up solutions to the following exercises.

1. Let $(\mathbb{Q}, +)$ be the group of rational numbers under addition, (\mathbb{Q}^*, \cdot) the group of non-zero rational numbers under multiplication, and (\mathbb{Q}^+, \cdot) the group of positive rational numbers under multiplication.
 - (a) Show that $(\mathbb{Q}^*, \cdot) \cong (\mathbb{Q}^+, \cdot) \times C_2$. [Hint: Recall that $C_2 = \{e^{0\pi i}, e^{\pi i}\} = \{1, -1\}$.]
 - (b) Describe the quotient groups $(\mathbb{Q}, +)/\langle -1 \rangle$ and $(\mathbb{Q}^*, \cdot)/\langle -1 \rangle$. In particular, what do the elements (cosets) look like?
 - (c) Use the Fundamental Homomorphism Theorem to prove that $(\mathbb{Q}^*, \cdot)/\langle -1 \rangle \cong (\mathbb{Q}^+, \cdot)$.
2. For Parts (a)–(d), a group G is given together with a normal subgroup H . Illustrate the embedding $\phi: H \rightarrow G$, and the quotient map $q: G \rightarrow G/H$, chained together so that $\text{im}(\phi) = \ker(q)$. An example for $G = \mathbb{Z}_6$ and $H = \mathbb{Z}_2$ is shown below:



- (a) $G = \mathbb{Z}_6$, $H = \mathbb{Z}_3$,
- (b) $G = D_3$, $H = C_3$,
- (c) $G = A_4$, $H = V_4$,
- (d) $G = S_n$, $H = A_n$ [don't draw the actual Cayley graphs for this one, just the maps].

Now, answer each of the following questions about each of your answers to Parts (a)–(d).

- (e) What map θ into H would satisfy the equation $\text{im } \theta = \ker \phi$? Choose one with the smallest possible domain.
- (f) What map θ' from G/H would satisfy the equation $\text{im}(q) = \ker(\theta')$? Choose one with the smallest possible codomain.
- (g) Add the two maps θ and θ' to your illustration.
- (h) The new chain of four homomorphisms is called a *short exact sequence*. It is one way to use homomorphisms to illustrate quotients, and it shows a connection between embeddings and quotient maps. Given a normal subgroup $H \triangleleft G$, show how to create a short exact sequence involving G and H .

3. Let A and B be normal subgroups of G . In this problem, you will prove the *Diamond Isomorphism Theorem*.
- Prove that $AB := \{ab : a \in A, b \in B\}$ and $BA := \{ba : a \in A, b \in B\}$ are equal as sets.
 - Prove that AB is a subgroup of G .
 - Prove that $B \triangleleft AB$ and $A \cap B \triangleleft A$.
 - Prove that $A/(A \cap B) \cong AB/B$. [*Hint*: Construct a homomorphism $\phi: A \rightarrow AB/B$ that has kernel $A \cap B$, then apply the FHT.]
 - Draw a diagram, or lattice, of G and its subgroups AB , A , B , and $A \cap B$. Interpret the result in Part (c) in terms of this diagram.
4. For each part below, consider the group $G = \langle A, B \rangle$ generated by the two matrices given. Assume that matrix multiplication is the binary operation, and $i = \sqrt{-1}$. To what common group is G isomorphic? Write down an explicit isomorphism (you only need to define it for the generators), and a group presentation for G .

$$(a) \quad A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$(b) \quad A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

$$(c) \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

5. In this exercise, you will prove that if A and B are normal subgroups and $AB = G$, then

$$G/(A \cap B) \cong (G/A) \times (G/B).$$

- (a) Consider the following map:

$$\phi: AB \longrightarrow (G/A) \times (G/B), \quad \phi(g) = (gA, gB).$$

Show that ϕ is a homomorphism.

- (b) Show that ϕ is surjective. That is, given any (g_1A, g_2B) , show that there is some $g = ab \in AB$ such that $\phi(g) = (g_1A, g_2B)$. [*Hint*: Try $g = a_2b_1$.]
- (c) Find $\ker(\phi)$ [you need to prove your answer is correct] and then apply the Fundamental Homomorphism Theorem.
6. For each order given below, list all abelian groups of that order by writing each one as a product of cyclic groups of prime power order. Additionally, write each one as a product of cyclic groups organized by “elementary divisors.”

- (a) 8
- (b) 54
- (c) 400
- (d) p^2q , where p and q are distinct primes.