Read the following, which can all be found either in the textbook or on the course website.

- Chapters 9.2–9.4 of *Visual Group Theory* (VGT).
- VGT Exercises 9.1, 9.4, 9.12, 9.14, 9.15, 9.19–9.27.

Write up solutions to the following exercises.

1. Let S be the following set of 7 "binary squares":

$$S = \left\{ \begin{array}{|cc|}\hline 0 & 0 \\ 0 & 0 \\ \hline\end{array}\,,\ \begin{array}{|cc|}\hline 0 & 1 \\ 1 & 0 \\ \hline\end{array}\,,\ \begin{array}{|cc|}\hline 1 & 0 \\ 0 & 1 \\ \hline\end{array}\,,\ \begin{array}{|cc|}\hline 1 & 1 \\ 0 & 0 \\ \hline\end{array}\,,\ \begin{array}{|cc|}\hline 0 & 1 \\ 0 & 1 \\ \hline\end{array}\,,\ \begin{array}{|cc|}\hline 0 & 0 \\ 1 & 1 \\ \hline\end{array}\,,\ \begin{array}{|cc|}\hline 1 & 0 \\ 1 & 0 \\ \hline\end{array} \right\}$$

   (a) Consider the (right) action of the group $G = V_4 = \langle v, h \rangle$ on S, where $\phi(v)$ reflects each square vertically, and $\phi(h)$ reflects each square horizontally. Draw an action diagram and compute the stabilizer of each element.

   (b) Consider the (right) action of the group $G = C_4 = \langle r \mid r^4 = e \rangle$ on S, where $\phi(r)$ rotates each square 90° clockwise. Draw an action diagram and compute the stabilizer of each element.

   (c) Suppose a group G of order 15 acts on S. Prove that there must be a fixed point.

2. Let $G = S_4$ act on itself by conjugation via the homomorphism

$$\phi \colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

   (a) How many orbits are there? Describe them as specifically as you can.

   (b) Find the orbit and the stabilizer of the following elements:

      i. e
      ii. $(1\ 2)$
      iii. $(1\ 2\ 3)$
      iv. $(1\ 2\ 3\ 4)$
      v. $(1\ 2)\,(3\ 4)$

3. A *p-group* is a group of order p^k for some integer k. Recall that the *center* of a group G is the set of all elements that commute with everything:

$$Z(G) = \{z \in G \mid gz = zg,\ \forall g \in G\} = \{z \in G \mid g^{-1}zg = z,\ \forall g \in G\}\,.$$

   Finally, a group G is *simple* if its only normal subgroups are G and $\langle e \rangle$.

   (a) Let G act on itself by conjugation via the homomorphism

$$\phi \colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

   Prove that $\mathrm{Fix}(\phi) = Z(G)$.

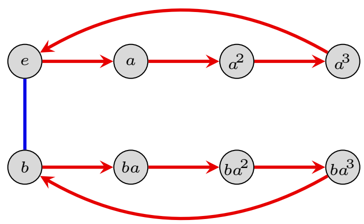
   (b) Prove that if G is a p-group, then $|Z(G)| > 1$. [*Hint*: Revisit the Class Equation.]

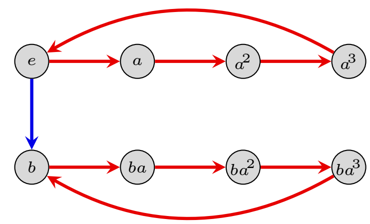
   (c) Use the result of the previous part to classify all simple p-groups.

4. Let G be an unknown group of order 8. By the First Sylow Theorem, G must contain a subgroup H of order 4.

   (a) If all subgroups of G of order 4 are isomorphic to V_4, then what group must G be? Completely justify your answer.

   (b) Next, suppose that G has a subgroup $H \cong C_4$. Then G has a Cayley diagram like one of the following:





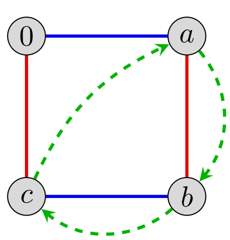
   Find all possibilities for finishing the Cayley diagram.

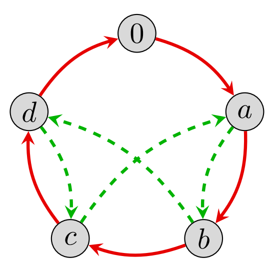
   (c) Label each completed Cayley diagram by isomorphism type. Justify your answer.

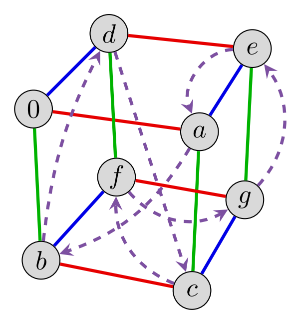
   (d) Make a complete list of all groups of order 8, up to isomorphism.

5. A *field* is a set $\mathbb{F}$ containing 0 and 1 that is an abelian group under addition, and (upon removing 0) an abelian group under multiplication, for which the distributative law holds. Common examples of fields are $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.

   There is a unique *finite* field $\mathbb{F}_q$ of order $q = p^k$ for every prime p and positive integer k. For all other $q \in \mathbb{N}$, there is no finite field of order q. For each of the fields $\mathbb{F}_4$, $\mathbb{F}_5$, and $\mathbb{F}_8$, the Cayley diagrams for addition and multiplication are shown below, overlayed on the same set of nodes. The solid arrows are the Cayley diagrams for addition and the dashed arrows are the Cayley diagrams for multiplication.







   (a) For each field above, determine whether or not the addition and multiplication operations are in fact, addition and multiplication modulo some number. If yes, relabel the vertices accordingly. If no, explain why it fails.

   (b) Create Cayley diagrams for the finite fields $\mathbb{F}_3$ and $\mathbb{F}_7$.