

- For each of the following rings R , determine the zero divisors (right and left, if appropriate), and the set $U(R)$ of units.
 - The set \mathcal{C}^1 of continuous real-valued functions $f: \mathbb{R} \rightarrow \mathbb{R}$.
 - The polynomial ring $\mathbb{R}[x]$.
 - $\mathbb{Z} \times \mathbb{Z}$, where addition and multiplication are defined componentwise.
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- Prove that if a left ideal I of a ring R contains a unit, then $I = R$.
- Let I and J be ideals of a ring R .
 - Prove that $I + J$, $I \cap J$, and IJ are ideals of R .
 - If R is commutative, then the set $(I : J) = \{r \in R \mid rJ \subseteq I\}$ is called the *ideal quotient* or *colon ideal* of I and J . Show that $(I : J)$ is an ideal of R .
 - Consider the ideals $I = 4\mathbb{Z}$ and $J = 6\mathbb{Z}$ of the ring $R = \mathbb{Z}$. Compute $I + J$, $I \cap J$, IJ , $(I : J)$, and $(J : I)$.
 - Repeat Part (c) for the ideals $I = m\mathbb{Z}$ and $J = n\mathbb{Z}$ of $R = \mathbb{Z}$.
- The left ideal generated by $X \subseteq R$ is defined as

$$(X) := \bigcap \{I \mid I \text{ is a left ideal s.t. } X \subseteq I \subseteq R\}.$$

- Prove that the left ideal generated by X is

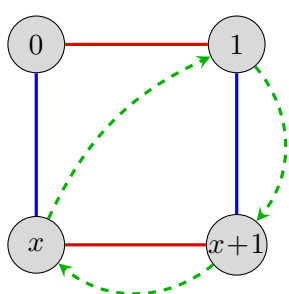
$$(X) = \{r_1x_1 + \cdots + r_nx_n \mid n \in \mathbb{N}, r_i \in R, x_i \in X\}.$$

- The two-sided ideal generated by $X \subseteq R$ is defined by replacing “left” with “two-sided” in the definition above. Prove this is also equal to

$$\{r_1x_1s_1 + \cdots + r_nx_ns_n \mid n \in \mathbb{N}, r_i, s_i \in R, x_i \in X\}.$$

- Find a (non-commutative) ring R and a set X such that the left and two-sided ideals generated by X are different.

- The finite field \mathbb{F}_4 on 4 elements can be constructed as the quotient of the polynomial $\mathbb{Z}_2[x]$ by the ideal $I = (x^2 + x + 1)$ generated by the irreducible polynomial $x^2 + x + 1$. The figure below shows a Cayley diagram, and multiplication and addition tables for the finite field $\mathbb{Z}_2[x]/(x^2 + x + 1) \cong \mathbb{F}_4$.



+	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

×	1	x	$x+1$
1	1	x	$x+1$
x	x	$x+1$	1
$x+1$	$x+1$	1	x

- (a) Find a degree-3 polynomial $f \in \mathbb{Z}_2[x]$ that is irreducible over \mathbb{Z}_2 , and a degree-2 polynomial $g \in \mathbb{Z}_3[x]$ that is irreducible over \mathbb{Z}_3 . [*Hint*: Any polynomial with no roots in the “prime field” \mathbb{Z}_p will work.]
- (b) Construct Cayley diagrams, addition, and multiplication tables for the finite fields

$$\mathbb{F}_8 \cong \mathbb{Z}_2[x]/(f) \quad \text{and} \quad \mathbb{F}_9 \cong \mathbb{Z}_3[x]/(g).$$

6. Prove the Fundamental Homomorphism Theorem (FHT) for rings: If $\phi: R \rightarrow S$ is a ring homomorphism, then $\text{Ker } \phi$ is a two-sided ideal of R , and $R/\text{Ker } \phi \cong \text{Im } \phi$. You may assume the FHT for groups.