Lecture 5.1: Groups acting on sets

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Overview

Intuitively, a group action occurs when a group G "naturally permutes" a set S *of states*.

For example:

- The "Rubik's cube group" consists of the 4.3×10^{19} actions that *permutated* the 4.3×10^{19} configurations of the cube.
- The group D_4 consists of the 8 symmetries of the square. These symmetries are *actions* that *permuted* the 8 configurations of the square.

Group actions help us understand the interplay between the actual group of actions and sets of objects that they "rearrange."

There are many other examples of groups that "act on" sets of objects. We will see examples when the group and the set have different sizes.

There is a rich theory of group actions, and it can be used to prove many deep results in group theory.

Actions vs. configurations

The group D_4 can be thought of as the 8 symmetries of the square: 

There is a subtle but *important* distinction to make, between the actual 8 symmetries of the square, and the 8 configurations.

For example, the 8 symmetries (alternatively, "actions") can be thought of as

$$e, \qquad r, \qquad r^2, \qquad r^3, \qquad f, \qquad rf, \qquad r^2 f, \qquad r^3 f\,.$$

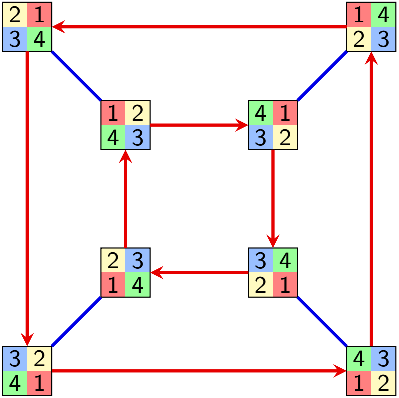
The 8 configurations (or *states*) of the square are the following:

When we were just learning about groups, we made an action diagram.

- The vertices correspond to the states.
- The edges correspond to generators.
- The paths corresponded to actions (group elements).

Actions diagrams

Here is the action diagram of the group $D_4 = \langle r, f \rangle$:



In the beginning of this course, we picked a configuration to be the "solved state," and this gave us a bijection between configurations and actions (group elements).

The resulting diagram was a Cayley diagram. In this chapter, we'll skip this step.

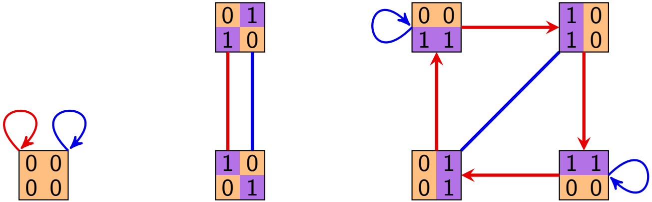
Actions diagrams

In all of the examples we saw in the beginning of the course, we had a bijective correspondence between actions and states. *This need not always happen!*

Suppose we have a size-7 set consisting of the following "binary squares."

$$S = \left\{ \begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&1\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}, \begin{bmatrix}1&0\\1&0\end{bmatrix} \right\}$$

The group $D_4 = \langle r, f \rangle$ "acts on S" as follows:



The action diagram above has some properties of Cayley diagrams, but there are some fundamental differences as well.

A "group switchboard"

Suppose we have a "switchboard" for G, with every element $g \in G$ having a "button."

If $a \in G$, then pressing the a-button rearranges the objects in our set S. In fact, it is a permutation of S; call it $\phi(a)$.

If $b \in G$, then pressing the b-button rearranges the objects in S a different way. Call this permutation $\phi(b)$.

The element $ab \in G$ also has a button. We require that pressing the ab-button yields the same result as pressing the a-button, followed by the b-button. That is,

$$\phi(ab) = \phi(a)\phi(b)\,, \qquad \text{for all } a, b \in G\,.$$

Let $\text{Perm}(S)$ be the group of permutations of S. Thus, if $|S| = n$, then $\text{Perm}(S) \cong S_n$. (We typically think of S_n as the permutations of $\{1, 2, \ldots, n\}$.)

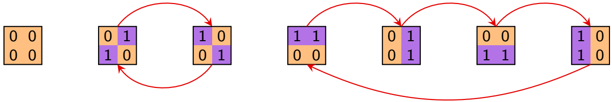
Definition

A group G acts on a set S if there is a homomorphism $\phi \colon G \to \text{Perm}(S)$.

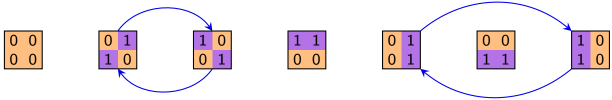
A "group switchboard"

Returning to our binary square example, pressing the r-button and f-button permutes the set S as follows:

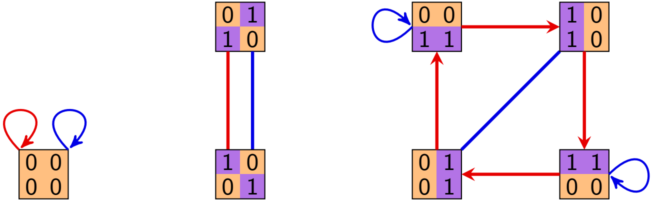
$\phi(r)$:



$\phi(f)$:



Observe how these permutations are encoded in the action diagram:



Left actions vs. right actions (an annoyance we can deal with)

As we've defined group actions, "*pressing the a-button followed by the b-button should be the same as pressing the ab-button*."

However, sometimes it has to be the same as "*pressing the ba-button*."

This is best seen by an example. Suppose our action is conjugation:

"Left group action"

$$\begin{tikzcd} H \arrow[r, "\text{conjugate by } a"] \arrow[rr, bend right, "\text{conjugate by } ba"'] & aHa^{-1} \arrow[r, "\text{conjugate by } b"] & baHa^{-1}b^{-1} \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ba)$$

"Right group action"

$$\begin{tikzcd} H \arrow[r, "\text{conjugate by } a"] \arrow[rr, bend right, "\text{conjugate by } ab"'] & a^{-1}Ha \arrow[r, "\text{conjugate by } b"] & b^{-1}a^{-1}Hab \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ab)$$

Some books forgo our "ϕ-notation" and use the following notation to distinguish left vs. right group actions:

$$g.(h.s) = (gh).s\,, \qquad\qquad (s.g).h = s.(gh)\,.$$

We'll usually keep the ϕ, and write $\phi(g)\phi(h)s = \phi(gh)s$ and $s.\phi(g)\phi(h) = s.\phi(gh)$. As with groups, the "dot" will be optional.

Left actions vs. right actions (an annoyance we can deal with)

Alternative definition (other textbooks)

A right group action is a mapping

$$G \times S \longrightarrow S\,, \qquad (a,s) \longmapsto s.a$$

such that

- $s.(ab) = (s.a).b$, for all $a, b \in G$ and $s \in S$
- $s.e = s$, for all $s \in S$.

A left group action can be defined similarly.

Pretty much all of the theorems for left actions hold for right actions.

Usually if there is a left action, there is a related right action. We will usually use right actions, and we will write

$$s.\phi(g)$$

for "the element of S that the permutation $\phi(g)$ sends s to," i.e., where pressing the g-button sends s.

If we have a left action, we'll write $\phi(g).s$.

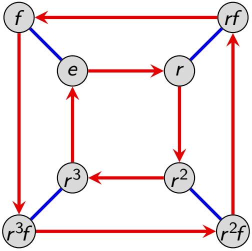
Cayley diagrams as action diagrams

Every Cayley diagram can be thought of as the action diagram of a particular (right) group action.

For example, consider the group $G = D_4 = \langle r, f \rangle$ acting on itself. That is, $S = D_4 = \{e, r, r^2, r^3, f, rf, r^2f, r^3f\}$.

Suppose that pressing the g-button on our "group switchboard" multiplies every element *on the right* by g.

Here is the action diagram:



We say that "*G acts on itself by right-multiplication.*"