Section 7: Ring theory

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Math 4120, Modern Algebra

What is a ring?

Definition

A ring is an additive (abelian) group R with an additional binary operation (multiplication), satisfying the distributive law:

$$x(y+z) = xy + xz \qquad \text{and} \qquad (y+z)x = yx + zx \quad \forall x, y, z \in R\,.$$

Remarks

- There need not be multiplicative inverses.
- Multiplication need not be commutative (it may happen that $xy \neq yx$).

A few more terms

If $xy = yx$ for all $x, y \in R$, then R is commutative.

If R has a multiplicative identity $1 = 1_R \neq 0$, we say that "R has identity" or "unity", or "R is a ring with 1."

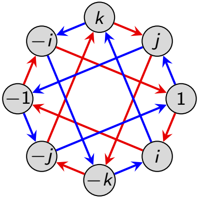
A subring of R is a subset $S \subseteq R$ that is also a ring.

What is a ring?

Examples

1. $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ are all commutative rings with 1.
2. $\mathbb{Z}_n$ is a commutative ring with 1.
3. For any ring R with 1, the set $M_n(R)$ of $n \times n$ matrices over R is a ring. It has identity $1_{M_n(R)} = I_n$ iff R has 1.
4. For any ring R, the set of functions $F = \{f \colon R \to R\}$ is a ring by defining
$$(f+g)(r) = f(r) + g(r), \qquad (fg)(r) = f(r)g(r)\,.$$
5. The set $S = 2\mathbb{Z}$ is a subring of $\mathbb{Z}$ but it does *not* have 1.
6. $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R} \right\}$ is a subring of $R = M_2(\mathbb{R})$. However, note that
$$1_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{but} \qquad 1_S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$
7. If R is a ring and x a variable, then the set
$$R[x] = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in R\}$$
is called the polynomial ring over R.

Another example: the quaternions



Recall the (unit) quaternion group:

$$Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = -1, \ ij = k \rangle.$$

Allowing addition makes them into a ring $\mathbb{H}$, called the quaternions, or Hamiltonians:

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}.$$

The set $\mathbb{H}$ is isomorphic to a subring of $M_4(\mathbb{R})$, the real-valued 4×4 matrices:

$$\mathbb{H} = \left\{ \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} \subseteq M_4(\mathbb{R}).$$

Formally, we have an embedding $\phi \colon \mathbb{H} \hookrightarrow M_4(\mathbb{R})$ where

$$\phi(i) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \phi(j) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \phi(k) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We say that $\mathbb{H}$ is represented by a set of matrices.

Units and zero divisors

Definition

Let R be a ring with 1. A unit is any $x \in R$ that has a multiplicative inverse. Let $U(R)$ be the set (a multiplicative group) of units of R.

An element $x \in R$ is a left zero divisor if $xy = 0$ for some $y \neq 0$. (Right zero divisors are defined analogously.)

Examples

1. Let $R = \mathbb{Z}$. The units are $U(R) = \{-1, 1\}$. There are no (nonzero) zero divisors.
2. Let $R = \mathbb{Z}_{10}$. Then 7 is a unit (and $7^{-1} = 3$) because $7 \cdot 3 = 1$. However, 2 is not a unit.
3. Let $R = \mathbb{Z}_n$. A nonzero $k \in \mathbb{Z}_n$ is a unit if $\gcd(n, k) = 1$, and a zero divisor if $\gcd(n, k) \geq 2$.
4. The ring $R = M_2(\mathbb{R})$ has zero divisors, such as:

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   The groups of units of $M_2(\mathbb{R})$ are the invertible matrices.

Group rings

Let R be a commutative ring (usually, $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$) and G a finite (multiplicative) group. We can define the group ring RG as

$$RG := \{a_1 g_1 + \cdots + a_n g_n \mid a_i \in R,\ g_i \in G\},$$

where multiplication is defined in the "obvious" way.

For example, let $R = \mathbb{Z}$ and $G = D_4 = \langle r, f \mid r^4 = f^2 = rfrf = 1 \rangle$, and consider the elements $x = r + r^2 - 3f$ and $y = -5r^2 + rf$ in $\mathbb{Z}D_4$. Their sum is

$$x + y = r - 4r^2 - 3f + rf,$$

and their product is

$$\begin{aligned} xy &= (r + r^2 - 3f)(-5r^2 + rf) = r(-5r^2 + rf) + r^2(-5r^2 + rf) - 3f(-5r^2 + rf) \\ &= -5r^3 + r^2 f - 5r^4 + r^3 f + 15fr^2 - 3frf = -5 - 8r^3 + 16r^2 f + r^3 f. \end{aligned}$$

Remarks

- The (real) Hamiltonians $\mathbb{H}$ is *not* the same ring as $\mathbb{R}Q_8$.
- If $g \in G$ has finite order $|g| = k > 1$, then RG always has zero divisors:

$$(1 - g)(1 + g + \cdots + g^{k-1}) = 1 - g^k = 1 - 1 = 0.$$

- RG contains a subring isomorphic to R, and the group of units $U(RG)$ contains a subgroup isomorphic to G.

Types of rings

Definition

If all nonzero elements of R have a multiplicative inverse, then R is a division ring. (Think: "field without commutativity".)

An integral domain is a commutative ring with 1 and with no (nonzero) zero divisors. (Think: "field without inverses".)

A field is just a commutative division ring. Moreover:

$$\text{fields} \subsetneq \text{division rings}$$

$$\text{fields} \subsetneq \text{integral domains} \subsetneq \text{all rings}$$

Examples

- Rings that are not integral domains: $\mathbb{Z}_n$ (composite n), $2\mathbb{Z}$, $M_n(\mathbb{R})$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{H}$.
- Integral domains that are not fields (or even division rings): $\mathbb{Z}$, $\mathbb{Z}[x]$, $\mathbb{R}[x]$, $\mathbb{R}[[x]]$ (formal power series).
- Division ring but not a field: $\mathbb{H}$.

Cancellation

When doing basic algebra, we often take for granted basic properties such as cancellation: $ax = ay \implies x = y$. However, *this need not hold in all rings!*

Examples where cancellation fails

- In $\mathbb{Z}_6$, note that $2 = 2 \cdot 1 = 2 \cdot 4$, but $1 \neq 4$.
- In $M_2(\mathbb{R})$, note that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.

However, everything works fine as long as there aren't any (nonzero) zero divisors.

Proposition

Let R be an integral domain and $a \neq 0$. If $ax = ay$ for some $x, y \in R$, then $x = y$.

Proof

If $ax = ay$, then $ax - ay = a(x - y) = 0$.

Since $a \neq 0$ and R has no (nonzero) zero divisors, then $x - y = 0$. $\square$

Finite integral domains

Lemma (HW)

If R is an integral domain and $0 \neq a \in R$ and $k \in \mathbb{N}$, then $a^k \neq 0$. $\square$

Theorem

Every finite integral domain is a field.

Proof

Suppose R is a finite integral domain and $0 \neq a \in R$. It suffices to show that a has a multiplicative inverse.

Consider the infinite sequence $a, a^2, a^3, a^4, \dots$, which must repeat.

Find $i > j$ with $a^i = a^j$, which means that

$$0 = a^i - a^j = a^j(a^{i-j} - 1).$$

Since R is an integral domain and $a^j \neq 0$, then $a^{i-j} = 1$.

Thus, $a \cdot a^{i-j-1} = 1$. $\square$

Ideals

In the theory of groups, we can quotient out by a subgroup if and only if it is a normal subgroup. The analogue of this for rings are (two-sided) ideals.

Definition

A subring $I \subseteq R$ is a left ideal if

$$rx \in I \qquad \text{for all } r \in R \text{ and } x \in I.$$

Right ideals, and two-sided ideals are defined similarly.

If R is commutative, then all left (or right) ideals are two-sided.

We use the term ideal and two-sided ideal synonymously, and write $I \trianglelefteq R$.

Examples

- $n\mathbb{Z} \trianglelefteq \mathbb{Z}$.
- If $R = M_2(\mathbb{R})$, then $I = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} : a, c \in \mathbb{R} \right\}$ is a left, but *not* a right ideal of R.
- The set $\text{Sym}_n(\mathbb{R})$ of symmetric $n \times n$ matrices is a subring of $M_n(\mathbb{R})$, but *not* an ideal.

Ideals

Remark

If an ideal I of R contains 1, then $I = R$.

Proof

Suppose $1 \in I$, and take an arbitrary $r \in R$.

Then $r1 \in I$, and so $r1 = r \in I$. Therefore, $I = R$. $\square$

It is not hard to modify the above result to show that if I contains *any* unit, then $I = R$. (HW)

Let's compare the concept of a normal subgroup to that of an ideal:

- normal subgroups are characterized by being invariant under conjugation:

$$H \leq G \text{ is normal iff } ghg^{-1} \in H \text{ for all } g \in G,\ h \in H.$$

- (left) ideals of rings are characterized by being invariant under (left) multiplication:

$$I \subseteq R \text{ is a (left) ideal iff } ri \in I \text{ for all } r \in R,\ i \in I.$$

Ideals generated by sets

Definition

The left ideal generated by a set $X \subset R$ is defined as:

$$(X) := \bigcap \left\{ I : \ I \text{ is a left ideal s.t. } X \subseteq I \subseteq R \right\}.$$

This is the smallest left ideal containing X.

There are analogous definitions by replacing "left" with "right" or "two-sided".

Recall the two ways to define the subgroup $\langle X \rangle$ generated by a subset $X \subseteq G$:

- "*Bottom up*": As the set of all finite products of elements in X;
- "*Top down*": As the intersection of all subgroups containing X.

Proposition (HW)

Let R be a ring *with unity*. The (left, right, two-sided) ideal generated by $X \subseteq R$ is:

- Left: $\{r_1x_1 + \cdots + r_nx_n : n \in \mathbb{N},\ r_i \in R,\ x_i \in X\}$,
- Right: $\{x_1r_1 + \cdots + x_nr_n : n \in \mathbb{N},\ r_i \in R,\ x_i \in X\}$,
- Two-sided: $\{r_1x_1s_1 + \cdots + r_nx_ns_n : n \in \mathbb{N},\ r_i, s_i \in R,\ x_i \in X\}$.

Ideals and quotients

Since an ideal I of R is an additive subgroup (and hence normal), then:

- $R/I = \{x + I \mid x \in R\}$ is the set of cosets of I in R;
- R/I is a quotient group; with the binary operation (addition) defined as

$$(x + I) + (y + I) := x + y + I.$$

It turns out that if I is also a two-sided ideal, then we can make R/I into a ring.

Proposition

If $I \subseteq R$ is a (two-sided) ideal, then R/I is a ring (called a quotient ring), where multiplication is defined by

$$(x + I)(y + I) := xy + I\,.$$

Proof

We need to show this is well-defined. Suppose $x + I = r + I$ and $y + I = s + I$. This means that $x - r \in I$ and $y - s \in I$.

It suffices to show that $xy + I = rs + I$, or equivalently, $xy - rs \in I$:

$$xy - rs = xy - ry + ry - rs = (x - r)y + r(y - s) \in I\,.$$

Finite fields

We've already seen that $\mathbb{Z}_p$ is a field if p is prime, and that finite integral domains are fields. But *what do these "other" finite fields look like?*

Let $R = \mathbb{Z}_2[x]$ be the polynomial ring over the field $\mathbb{Z}_2$. (Note: we can ignore all negative signs.)

The polynomial $f(x) = x^2 + x + 1$ is irreducible over $\mathbb{Z}_2$ because it does not have a root. (Note that $f(0) = f(1) = 1 \neq 0$.)

Consider the ideal $I = (x^2 + x + 1)$, the set of multiples of $x^2 + x + 1$.

In the quotient ring R/I, we have the relation $x^2 + x + 1 = 0$, or equivalently, $x^2 = -x - 1 = x + 1$.

The quotient has only 4 elements:

$$0 + I, \qquad 1 + I, \qquad x + I, \qquad (x+1) + I.$$

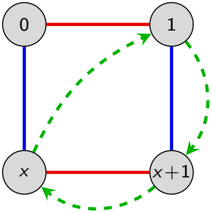
As with the quotient group (or ring) $\mathbb{Z}/n\mathbb{Z}$, we usually drop the "I", and just write

$$R/I = \mathbb{Z}_2[x]/(x^2 + x + 1) \cong \{0,\ 1,\ x,\ x+1\}.$$

It is easy to check that this is a field!

Finite fields

Here is a Cayley diagram, and the operation tables for $R/I = \mathbb{Z}_2[x]/(x^2+x+1)$:



+	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	1	x	$x+1$
1	1	x	$x+1$
x	x	$x+1$	1
$x+1$	$x+1$	1	x

Theorem

There exists a finite field $\mathbb{F}_q$ of order q, which is unique up to isomorphism, iff $q = p^n$ for some prime p. If $n > 1$, then this field is isomorphic to the quotient ring

$$\mathbb{Z}_p[x]/(f)\,,$$

where f is *any* irreducible polynomial of degree n.

Much of the error correcting techniques in coding theory are built using mathematics over $\mathbb{F}_{2^8} = \mathbb{F}_{256}$. This is what allows your CD to play despite scratches.

Motivation (spoilers!)

Many of the big ideas from group homomorphisms carry over to ring homomorphisms.

Group theory

- The quotient group G/N exists iff N is a normal subgroup.
- A homomorphism is a structure-preserving map: $f(x * y) = f(x) * f(y)$.
- The kernel of a homomorphism is a normal subgroup: $\operatorname{Ker}\phi \trianglelefteq G$.
- For every normal subgroup $N \trianglelefteq G$, there is a natural quotient homomorphism $\phi\colon G \to G/N$, $\phi(g) = gN$.
- There are four standard isomorphism theorems for groups.

Ring theory

- The quotient ring R/I exists iff I is a two-sided ideal.
- A homomorphism is a structure-preserving map: $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$.
- The kernel of a homomorphism is a two-sided ideal: $\operatorname{Ker}\phi \trianglelefteq R$.
- For every two-sided ideal $I \trianglelefteq R$, there is a natural quotient homomorphism $\phi\colon R \to R/I$, $\phi(r) = r + I$.
- There are four standard isomorphism theorems for rings.

Ring homomorphisms

Definition

A ring homomorphism is a function $f\colon R \to S$ satisfying

$$f(x+y) = f(x) + f(y) \qquad \text{and} \qquad f(xy) = f(x)f(y) \quad \text{for all } x, y \in R.$$

A ring isomorphism is a homomorphism that is bijective.

The kernel $f\colon R \to S$ is the set $\operatorname{Ker} f := \{x \in R : f(x) = 0\}$.

Examples

1. The function $\phi\colon \mathbb{Z} \to \mathbb{Z}_n$ that sends $k \mapsto k \pmod{n}$ is a ring homomorphism with $\operatorname{Ker}(\phi) = n\mathbb{Z}$.
2. For a fixed real number $\alpha \in \mathbb{R}$, the "evaluation function"

   $$\phi\colon \mathbb{R}[x] \longrightarrow \mathbb{R}\,, \qquad\qquad \phi\colon p(x) \longmapsto p(\alpha)$$

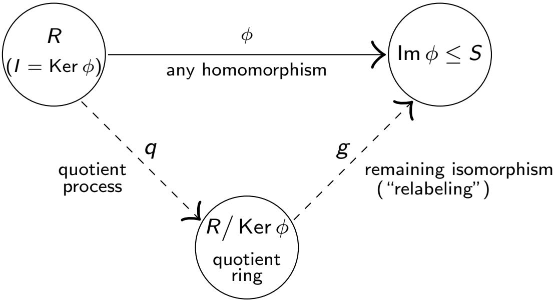
   is a homomorphism. The kernel consists of all polynomials that have α as a root.
3. The following is a homomorphism, for the ideal $I = (x^2 + x + 1)$ in $\mathbb{Z}_2[x]$:

   $$\phi\colon \mathbb{Z}_2[x] \longrightarrow \mathbb{Z}_2[x]/I, \qquad\qquad f(x) \longmapsto f(x) + I\,.$$

The isomorphism theorems for rings

Fundamental homomorphism theorem

If $\phi\colon R \to S$ is a ring homomorphism, then $\operatorname{Ker}\phi$ is an ideal and $\operatorname{Im}(\phi) \cong R/\operatorname{Ker}(\phi)$.



Proof (HW)

The statement holds for the underlying additive group R. Thus, it remains to show that $\operatorname{Ker}\phi$ is a (two-sided) ideal, and the following map is a ring homomorphism:

$$g\colon R/I \longrightarrow \operatorname{Im}\phi\,, \qquad\qquad g(x+I) = \phi(x)\,.$$

The second isomorphism theorem for rings

Suppose S is a subring and I an ideal of R. Then

(i) The sum $S+I=\{s+i \mid s\in S,\ i\in I\}$ is a subring of R and the intersection $S\cap I$ is an ideal of S.

(ii) The following quotient rings are isomorphic:

$$(S+I)/I \cong S/(S\cap I)\,.$$

$$\begin{tikzcd}[column sep=small]
& R \arrow[d, no head] & \\
& S+I \arrow[dl, no head] \arrow[dr, no head] & \\
S \arrow[dr, no head] & & I \arrow[dl, no head] \\
& S\cap I &
\end{tikzcd}$$

Proof (sketch)

$S+I$ is an additive subgroup, and it's closed under multiplication because

$$s_1,s_2\in S,\ i_1,i_2\in I \quad\Longrightarrow\quad (s_1+i_1)(s_2+i_2)=\underbrace{s_1s_2}_{\in S}+\underbrace{s_1i_2+i_1s_2+i_1i_2}_{\in I}\in S+I.$$

Showing $S\cap I$ is an ideal of S is straightforward (homework exercise).

We already know that $(S+I)/I\cong S/(S\cap I)$ as additive groups.

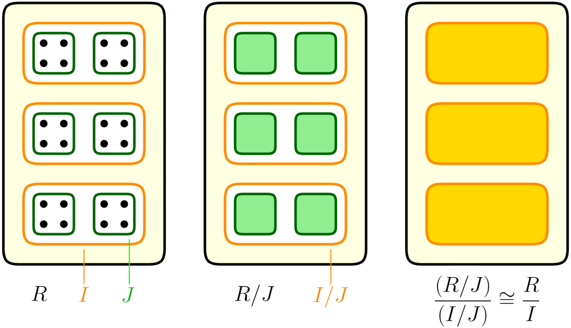
One explicit isomorphism is $\phi\colon s+(S\cap I)\mapsto s+I$. It is easy to check that $\phi\colon 1\mapsto 1$ and ϕ preserves products. $\square$

The third isomorphism theorem for rings

Freshman theorem

Suppose R is a ring with ideals $J \subseteq I$. Then I/J is an ideal of R/J and

$$(R/J)/(I/J) \cong R/I\,.$$

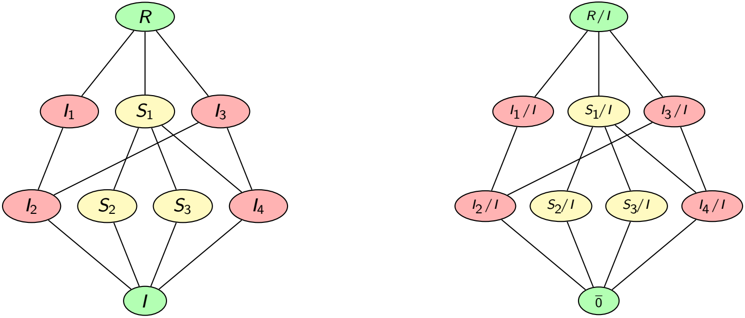


(Thanks to Zach Teitler of Boise State for the concept and graphic!)

The fourth isomorphism theorem for rings

Correspondence theorem

Let I be an ideal of R. There is a bijective correspondence between subrings (& ideals) of R/I and subrings (& ideals) of R that contain I. In particular, every ideal of R/I has the form J/I, for some ideal J satisfying $I \subseteq J \subseteq R$.



subrings & ideals that contain I

subrings & ideals of R/I

Maximal ideals

Definition

An ideal I of R is maximal if $I \neq R$ and if $I \subseteq J \subseteq R$ holds for some ideal J, then $J = I$ or $J = R$.

A ring R is simple if its only (two-sided) ideals are 0 and R.

Examples

1. If $n \neq 0$, then the ideal $M = (n)$ of $R = \mathbb{Z}$ is maximal if and only if n is prime.
2. Let $R = \mathbb{Q}[x]$ be the set of all polynomials over $\mathbb{Q}$. The ideal $M = (x)$ consisting of all polynomials with constant term zero is a maximal ideal.

   Elements in the quotient ring $\mathbb{Q}[x]/(x)$ have the form $f(x) + M = a_0 + M$.
3. Let $R = \mathbb{Z}_2[x]$, the polynomials over $\mathbb{Z}_2$. The ideal $M = (x^2 + x + 1)$ is maximal, and $R/M \cong \mathbb{F}_4$, the (unique) finite field of order 4.

In all three examples above, the quotient R/M is a field.

Maximal ideals

Theorem

Let R be a commutative ring with 1. The following are equivalent for an ideal $I \subseteq R$.

(i) I is a maximal ideal;

(ii) R/I is simple;

(iii) R/I is a field.

Proof

The equivalence (i)$\Leftrightarrow$(ii) is immediate from the Correspondence Theorem.

For (ii)$\Leftrightarrow$(iii), we'll show that an *arbitrary* ring R is simple iff R is a field.

"$\Rightarrow$": Assume R is simple. Then $(a) = R$ for any nonzero $a \in R$.

Thus, $1 \in (a)$, so $1 = ba$ for some $b \in R$, so $a \in U(R)$ and R is a field. ✓

"$\Leftarrow$": Let $I \subseteq R$ be a nonzero ideal of a field R. Take any nonzero $a \in I$.

Then $a^{-1}a \in I$, and so $1 \in I$, which means $I = R$. ✓ □

Prime ideals

Definition

Let R be a commutative ring. An ideal $P \subset R$ is prime if $ab \in P$ implies either $a \in P$ or $b \in P$.

Note that $p \in \mathbb{N}$ is a prime number iff $p = ab$ implies either $a = p$ or $b = p$.

Examples

1. The ideal (n) of $\mathbb{Z}$ is a prime ideal iff n is a prime number (possibly $n = 0$).
2. In the polynomial ring $\mathbb{Z}[x]$, the ideal $I = (2, x)$ is a prime ideal. It consists of all polynomials whose constant coefficient is even.

Theorem

An ideal $P \subseteq R$ is prime iff R/P is an integral domain.

The proof is straightforward (HW). Since fields are integral domains, the following is immediate:

Corollary

In a commutative ring, every maximal ideal is prime.

Divisibility and factorization

A ring is in some sense, a generalization of the familiar number systems like $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, where we are allowed to add, subtract, and multiply.

Two key properties about these structures are:

- multiplication is commutative,
- there are no (nonzero) zero divisors.

Blanket assumption

Throughout this lecture, unless explicitly mentioned otherwise, R is assumed to be an integral domain, and we will define $R^* := R \setminus \{0\}$.

The integers have several basic properties that we usually take for granted:

- every nonzero number can be factored uniquely into primes;
- any two numbers have a unique greatest common divisor and least common multiple;
- there is a Euclidean algorithm, which can find the gcd of two numbers.

Surprisingly, these need not always hold in integrals domains! We would like to understand this better.

Divisibility

Definition

If $a, b \in R$, say that a divides b, or b is a multiple of a if $b = ac$ for some $c \in R$. We write $a \mid b$.

If $a \mid b$ and $b \mid a$, then a and b are associates, written $a \sim b$.

Examples

- In $\mathbb{Z}$: n and $-n$ are associates.
- In $\mathbb{R}[x]$: $f(x)$ and $c \cdot f(x)$ are associates for any $c \neq 0$.
- The only associate of 0 is itself.
- The associates of 1 are the units of R.

Proposition (HW)

Two elements $a, b \in R$ are associates if and only if $a = bu$ for some unit $u \in U(R)$.

This defines an equivalence relation on R, and partitions R into equivalence classes.

Irreducibles and primes

Note that units divide everything: if $b \in R$ and $u \in U(R)$, then $u \mid b$.

Definition

If $b \in R$ is not a unit, and the only divisors of b are units and associates of b, then b is irreducible.

An element $p \in R$ is prime if p is not a unit, and $p \mid ab$ implies $p \mid a$ or $p \mid b$.

Proposition

If $0 \neq p \in R$ is prime, then p is irreducible.

Proof

Suppose p is prime but not irreducible. Then $p = ab$ with $a, b \notin U(R)$.

Then (wlog) $p \mid a$, so $a = pc$ for some $c \in R$. Now,

$$p = ab = (pc)b = p(cb)\,.$$

This means that $cb = 1$, and thus $b \in U(R)$, a contradiction. $\square$

Irreducibles and primes

Caveat: Irreducible $\not\Rightarrow$ prime

Consider the ring $R_{-5} := \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$.

$$3 \mid (2 + \sqrt{-5})(2 - \sqrt{-5}) = 9 = 3 \cdot 3\,,$$

but $3 \nmid 2 + \sqrt{-5}$ and $3 \nmid 2 - \sqrt{-5}$.

Thus, 3 is irreducible in R_{-5} but *not* prime.

When irreducibles fail to be prime, we can lose nice properties like unique factorization.

Things can get really bad: not even the *lengths* of factorizations into irreducibles need be the same!

For example, consider the ring $R = \mathbb{Z}[x^2, x^3]$. Then

$$x^6 = x^2 \cdot x^2 \cdot x^2 = x^3 \cdot x^3.$$

The element $x^2 \in R$ is not prime because $x^2 \mid x^3 \cdot x^3$ yet $x^2 \nmid x^3$ in R (note: $x \notin R$).

Principal ideal domains

Fortunately, there is a type of ring where such "bad things" don't happen.

Definition

An ideal I generated by a single element $a \in R$ is called a principal ideal. We denote this by $I = (a)$.

If every ideal of R is principal, then R is a principal ideal domain (PID).

Examples

The following are all PIDs (stated without proof):

- The ring of integers, $\mathbb{Z}$.
- Any field F.
- The polynomial ring $F[x]$ over a field.

As we will see shortly, PIDs are "nice" rings. Here are some properties they enjoy:

- pairs of elements have a "greatest common divisor" & "least common multiple";
- irreducible $\Rightarrow$ prime;
- Every element factors uniquely into primes.

Greatest common divisors & least common multiples

Proposition

If $I \subseteq \mathbb{Z}$ is an ideal, and $a \in I$ is its smallest positive element, then $I = (a)$.

Proof

Pick any positive $b \in I$. Write $b = aq + r$, for $q, r \in \mathbb{Z}$ and $0 \leq r < a$.

Then $r = b - aq \in I$, so $r = 0$. Therefore, $b = qa \in (a)$. $\square$

Definition

A common divisor of $a, b \in R$ is an element $d \in R$ such that $d \mid a$ and $d \mid b$.

Moreover, d is a greatest common divisor (GCD) if $c \mid d$ for all other common divisors c of a and b.

A common multiple of $a, b \in R$ is an element $m \in R$ such that $a \mid m$ and $b \mid m$.

Moreover, m is a least common multiple (LCM) if $m \mid n$ for all other common multiples n of a and b.

Nice properties of PIDs

Proposition

If R is a PID, then any $a, b \in R^*$ have a GCD, $d = \gcd(a, b)$.

It is *unique up to associates*, and can be written as $d = xa + yb$ for some $x, y \in R$.

Proof

Existence. The ideal generated by a and b is

$$I = (a, b) = \{ua + vb : u, v \in R\}.$$

Since R is a PID, we can write $I = (d)$ for some $d \in I$, and so $d = xa + yb$.

Since $a, b \in (d)$, both $d \mid a$ and $d \mid b$ hold.

If c is a divisor of a & b, then $c \mid xa + yb = d$, so d is a GCD for a and b. ✓

Uniqueness. If d' is another GCD, then $d \mid d'$ and $d' \mid d$, so $d \sim d'$. ✓ □

Nice properties of PIDs

Corollary

If R is a PID, then every irreducible element is prime.

Proof

Let $p \in R$ be irreducible and suppose $p \mid ab$ for some $a, b \in R$.

If $p \nmid a$, then $\gcd(p, a) = 1$, so we may write $1 = xa + yp$ for some $x, y \in R$. Thus

$$b = (xa + yp)b = x(ab) + (yb)p\,.$$

Since $p \mid x(ab)$ and $p \mid (yb)p$, then $p \mid x(ab) + (yb)p = b$. $\square$

Not surprisingly, least common multiples also have a nice characterization in PIDs.

Proposition (HW)

If R is a PID, then any $a, b \in R^*$ have an LCM, $m = \operatorname{lcm}(a, b)$.

It is *unique up to associates*, and can be characterized as a generator of the ideal $I := (a) \cap (b)$.

Unique factorization domains

Definition

An integral domain is a unique factorization domain (UFD) if:

(i) Every nonzero element is a product of irreducible elements;

(ii) Every irreducible element is prime.

Examples

1. $\mathbb{Z}$ is a UFD: Every integer $n \in \mathbb{Z}$ can be uniquely factored as a product of irreducibles (primes):
$$n = p_1^{d_1} p_2^{d_2} \cdots p_k^{d_k}.$$
This is the *fundamental theorem of arithmetic*.

2. The ring $\mathbb{Z}[x]$ is a UFD, because every polynomial can be factored into irreducibles. But it is not a PID because the following ideal is not principal:
$$(2, x) = \{f(x) : \text{ the constant term is even}\}.$$

3. The ring R_{-5} is not a UFD because $9 = 3 \cdot 3 = (2 + \sqrt{-5})(2 - \sqrt{-5})$.

4. We've shown that (ii) holds for PIDs. Next, we will see that (i) holds as well.

Unique factorization domains

Theorem

If R is a PID, then R is a UFD.

Proof

We need to show Condition (i) holds: every element is a product of irreducibles. A ring is Noetherian if every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

stabilizes, meaning that $I_k = I_{k+1} = I_{k+2} = \cdots$ holds for some k.

Suppose R is a PID. It is not hard to show that R is Noetherian (HW). Define

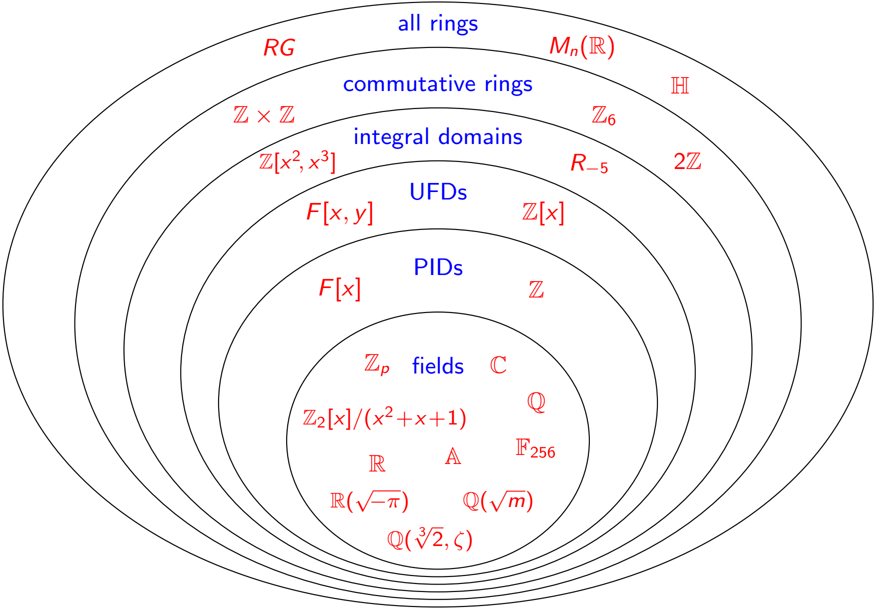
$$X = \{a \in R^* \setminus U(R) : a \text{ can't be written as a product of irreducibles}\}.$$

If $X \neq \emptyset$, then pick $a_1 \in X$. Factor this as $a_1 = a_2 b$, where $a_2 \in X$ and $b \notin U(R)$. Then $(a_1) \subsetneq (a_2) \subsetneq R$, and repeat this process. We get an ascending chain

$$(a_1) \subsetneq (a_2) \subsetneq (a_3) \subsetneq \cdots$$

that does not stabilize. This is impossible in a PID, so $X = \emptyset$. $\square$

Summary of ring types



The Euclidean algorithm

Around 300 B.C., Euclid wrote his famous book, the *Elements*, in which he described what is now known as the Euclidean algorithm:

Proposition VII.2 (Euclid's *Elements*)

Given two numbers not prime to one another, to find their greatest common measure.

The algorithm works due to two key observations:

- If $a \mid b$, then $\gcd(a, b) = a$;
- If $a = bq + r$, then $\gcd(a, b) = \gcd(b, r)$.

This is best seen by an example: Let $a = 654$ and $b = 360$.

$$\begin{aligned}
654 &= 360 \cdot 1 + 294 & \qquad \gcd(654, 360) &= \gcd(360, 294) \\
360 &= 294 \cdot 1 + 66 & \gcd(360, 294) &= \gcd(294, 66) \\
294 &= 66 \cdot 4 + 30 & \gcd(294, 66) &= \gcd(66, 30) \\
66 &= 30 \cdot 2 + 6 & \gcd(66, 30) &= \gcd(30, 6) \\
30 &= 6 \cdot 5 & \gcd(30, 6) &= 6.
\end{aligned}$$

We conclude that $\gcd(654, 360) = 6$.

Euclidean domains

Loosely speaking, a Euclidean domain is any ring for which the Euclidean algorithm still works.

Definition

An integral domain R is Euclidean if it has a degree function $d\colon R^* \to \mathbb{Z}$ satisfying:

(i) non-negativity: $d(r) \geq 0 \quad \forall r \in R^*$.

(ii) monotonicity: $d(a) \leq d(ab)$ for all $a, b \in R^*$.

(iii) division-with-remainder property: For all $a, b \in R$, $b \neq 0$, there are $q, r \in R$ such that

$$a = bq + r \qquad \text{with} \qquad r = 0 \quad \text{or} \quad d(r) < d(b)\,.$$

Note that Property (ii) could be restated to say: *If $a \mid b$, then $d(a) \leq d(b)$*;

Examples

- $R = \mathbb{Z}$ is Euclidean. Define $d(r) = |r|$.
- $R = F[x]$ is Euclidean if F is a field. Define $d(f(x)) = \deg f(x)$.
- The Gaussian integers $R_{-1} = \mathbb{Z}[\sqrt{-1}] = \{a + bi : a, b \in \mathbb{Z}\}$ is Euclidean with degree function $d(a + bi) = a^2 + b^2$.

Euclidean domains

Proposition

If R is Euclidean, then $U(R) = \{x \in R^* : d(x) = d(1)\}$.

Proof

$\subseteq$": First, we'll show that associates have the same degree. Take $a \sim b$ in R^*:

$$\begin{array}{lll} a \mid b & \Longrightarrow & d(a) \leq d(b) \\ b \mid a & \Longrightarrow & d(b) \leq d(a) \end{array} \qquad \Longrightarrow \qquad d(a) = d(b).$$

If $u \in U(R)$, then $u \sim 1$, and so $d(u) = d(1)$. ✓

"$\supseteq$": Suppose $x \in R^*$ and $d(x) = d(1)$.

Then $1 = qx + r$ for some $q \in R$ with either $r = 0$ or $d(r) < d(x) = d(1)$.

If $r \neq 0$, then $d(1) \leq d(r)$ since $1 \mid r$.

Thus, $r = 0$, and so $qx = 1$, hence $x \in U(R)$. ✓ □

Euclidean domains

Proposition

If R is Euclidean, then R is a PID.

Proof

Let $I \neq 0$ be an ideal and pick some $b \in I$ with $d(b)$ minimal.

Pick $a \in I$, and write $a = bq + r$ with either $r = 0$, or $d(r) < d(b)$.

This latter case is impossible: $r = a - bq \in I$, and by minimality, $d(b) \leq d(r)$.

Therefore, $r = 0$, which means $a = bq \in (b)$. Since a was arbitrary, $I = (b)$. $\square$

Exercises.

(i) The ideal $I = (3, 2 + \sqrt{-5})$ is not principal in R_{-5}.

(ii) If R is an integral domain, then $I = (x, y)$ is not principal in $R[x, y]$.

Corollary

The rings R_{-5} (not a PID or UFD) and $R[x, y]$ (not a PID) are not Euclidean.

Algebraic integers

The algebraic integers are the roots of *monic* polynomials in $\mathbb{Z}[x]$. This is a subring of the algebraic numbers (roots of all polynomials in $\mathbb{Z}[x]$).

Assume $m \in \mathbb{Z}$ is square-free with $m \neq 0, 1$. Recall the quadratic field

$$\mathbb{Q}(\sqrt{m}) = \left\{ p + q\sqrt{m} \mid p, q \in \mathbb{Q} \right\}.$$

Definition

The ring R_m is the set of algebraic integers in $\mathbb{Q}(\sqrt{m})$, i.e., the subring consisting of those numbers that are roots of monic quadratic polynomials $x^2 + cx + d \in \mathbb{Z}[x]$.

Facts

- R_m is an integral domain with 1.
- Since m is square-free, $m \not\equiv 0 \pmod 4$. For the other three cases:

$$R_m = \begin{cases} \mathbb{Z}[\sqrt{m}] = \left\{ a + b\sqrt{m} : a, b \in \mathbb{Z} \right\} & m \equiv 2 \text{ or } 3 \pmod 4 \\ \mathbb{Z}\left[\frac{1+\sqrt{m}}{2}\right] = \left\{ a + b\left(\frac{1+\sqrt{m}}{2}\right) : a, b \in \mathbb{Z} \right\} & m \equiv 1 \pmod 4 \end{cases}$$

- R_{-1} is the Gaussian integers, which is a PID. (easy)
- R_{-19} is a PID. (hard)

Algebraic integers

Definition

For $x = r + s\sqrt{m} \in \mathbb{Q}(\sqrt{m})$, define the norm of x to be

$$N(x) = (r + s\sqrt{m})(r - s\sqrt{m}) = r^2 - ms^2 .$$

R_m is norm-Euclidean if it is a Euclidean domain with $d(x) = |N(x)|$.

Note that the norm is multiplicative: $N(xy) = N(x)N(y)$.

Exercises

Assume $m \in \mathbb{Z}$ is square-free, with $m \neq 0, 1$.

- $u \in U(R_m)$ iff $|N(u)| = 1$.
- If $m \geq 2$, then $U(R_m)$ is infinite.
- $U(R_{-1}) = \{\pm 1, \pm i\}$ and $U(R_{-3}) = \{ \pm 1, \pm \frac{1 \pm \sqrt{-3}}{2} \}$.
- If $m = -2$ or $m < -3$, then $U(R_m) = \{\pm 1\}$.

Euclidean domains and algebraic integers

Theorem

R_m is norm-Euclidean iff

$$m \in \{-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\}.$$

Theorem (D.A. Clark, 1994)

The ring R_{69} is a Euclidean domain that is *not* norm-Euclidean.

Let $\alpha = (1+\sqrt{69})/2$ and $c > 25$ be an integer. Then the following degree function works for R_{69}, defined on the prime elements:

$$d(p) = \begin{cases} |N(p)| & \text{if } p \neq 10+3\alpha \\ c & \text{if } p = 10+3\alpha \end{cases}$$

Theorem

If $m < 0$ and $m \notin \{-11, -7, -3, -2, -1\}$, then R_m is not Euclidean.

Open problem

Classify which R_m's are PIDs, and which are Euclidean.

PIDs that are not Euclidean

Theorem

If $m < 0$, then R_m is a PID iff

$$m \in \{\underbrace{-1, -2, -3, -7, -11}_{\text{Euclidean}}, -19, -43, -67, -163\}.$$

Recall that R_m is norm-Euclidean iff

$$m \in \{-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\}.$$

Corollary

If $m < 0$, then R_m is a PID that is not Euclidean iff $m \in \{-19, -43, -67, -163\}$.

Algebraic integers

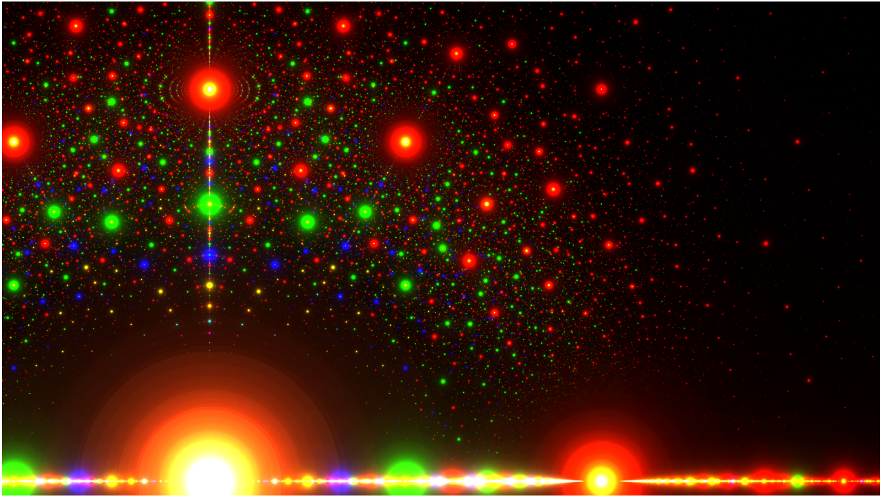


Figure: Algebraic numbers in the complex plane. Colors indicate the coefficient of the leading term: red $=1$ (algebraic integer), green $=2$, blue $=3$, yellow $=4$. Large dots mean fewer terms and smaller coefficients. Image from Wikipedia (made by Stephen J. Brooks).

Algebraic integers

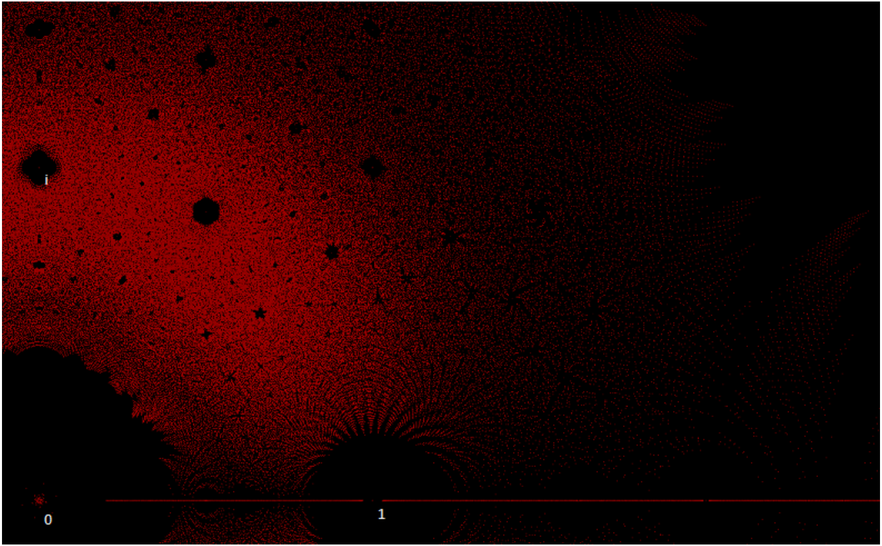


Figure: Algebraic integers in the complex plane. Each red dot is the root of a monic polynomial of degree ≤ 7 with coefficients from $\{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$. From Wikipedia.

Summary of ring types