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Assignment online_HW_10_number_theory due 06/07/2019 at 11:59pm EDT

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1. **(3 points)** `Library/SUNYSB/divisibility.pg`

Enter T or F depending on whether the statement is a true proposition or not. (You must enter T or F – True and False will not work.)

___1. 19 divides 267
___2. 209 divides 11
___3. 12065 is divisible by 116
___4. 12 is divisible by 156
___5. 300 | 20
___6. 11 divides 165
___7. 17 is divisible by 306
___8. 199 is divisible by 18
___9. 15 | 270

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2. **(5 points)** `Library/NewHampshire/unh_schoolib/Number_Theory/notsns201.pg`

1 is the only number that has just one positive divisor and we call it a *unit*. 1 divides every number.

3 is a number that has exactly two positive divisors. Its divisors are 1 and 3. A number with exactly two divisors is called a *prime*.

12 is a number with six divisors. Its divisors are 1,2,3,4,6,12. A number with more than two positive divisors is called *composite*.

The smallest number that is a prime is ___.

The smallest number that is composite is ___.

There are ___ primes that are less than 10.

There are ___ composites that are less than 10.

The smallest prime greater than 50 is ___

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3. **(4 points)** `Library/SDSU/Discrete/IntegersAndRationals/pL10.pg`

Suppose an integer x is divisible by 21.
What other positive integers must also divide x?
[Hint: Think about the transitivity of "divides." Enter your answer as a comma-separated list.]

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Suppose an integer a is divisible by 16.
What other positive integers must also divide a?
[Enter your answer as a comma-separated list.]

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4. **(5 points)** `Library/NewHampshire/unh_schoolib/GCF_LCM/gcfsne401.pg`

There is another neat way to use prime factorization. The *greatest common divisor* (also called greatest common factor) of two numbers is the largest number that divides each of them. For example, the greatest common divisor of 8 and 12 is 4. The *least common multiple* of two numbers is the smallest number that both of the numbers divide. The last common multiple of 8 and 12 is 24.

Now, when we write the prime factorizations of 8 and 12 we see that $8 = 2^3$ and $12 = (2^2) * 3$. Notice that for the prime 2, we have the third power in 8 and the second power in 12. Thus the second power of 2, that is, 4 must divide both numbers. Since the prime 3 does not occur in 8, it can't be a factor of the greatest common divisor. Thus the greatest common divisor of 2^3 and $(2^2) * 3$ is 2^2.

We think similarly to find the least common multiple of 2^3 and $(2^2) * 3$. For the least common multiple we need the highest power of 2 in either of the factors since that power of 2 must divide the least common multiple. Similarly we need the highest power of 3. Thus the least common multiple must be $(2^3) * 3$.

It is interesting to notice that the product of 12 and 8 is 96 and so is the product of gcd(12,8) and lcm(12,8). [Here we write gcd(12,8) for the greatest common divisor of 12 and 8 and lcm(12,8) for the least common multiple of 12 and 8. You must suspect that this is not a coincidence. Now let's see why that is true. We will write the equation using the prime factorizations.

$[(2^2) * 3] * [2^3] = [2^2][(2^3) * 3]$

Notice that the gcd takes the smaller powers of both 2 and 3 [formally we think of 2^3 as $(2^3) * (3^0)$ while the lcm takes the larger powers of 2 and 3 so everything in the first product has somewhere to go in the second product. This line of thinking tells us that the prouct of the lcm and gcd of two numbers will always be the product of the two numbers.

1) gcd(10,11)=___, while lcm(10,11)=___
2) gcd(12,20)=___, while lcm(12,20)=___
3) gcd(36,60)=___, while lcm(36,60)=___
4) gcd(45,75)=___, while lcm(45,75)=___
5) gcd(28800,138240)=___, while lcm(28800,138240)=___

You can probably see that when the numbers get large, even prime factorization is not terribly fast.

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5. **(3 points)** `Library/SDSU/Discrete/IntegersAndRationals/pL4.pg`

Assuming that p, q, r are distinct primes, how many positive divisors does m have in the following cases:

If $m = q^3$ ___

If $m = p^2 q^2$ ___

If $m = pqr$ ___

[Hint: Think of a particular number with the given factorization.]

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6. **(4 points)** `Library/NewHampshire/NECAP/grade11/gr11-2009/n11-2009-9s.pg`

Which expression must be divisible by 3 for every integer x?

- A. 3x+1
- B. 8x+6
- C. 4x-1
- D. 12x-9

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7. **(6 points)** `Library/SUNYSB/proofnumberth.pg`

Enter T or F depending on whether the statement is true or not. (You must enter T or F – True and False will not work.) If it is true do a proof and if it is false provide a counter-example (Of course the proofs and counter-examples must be written down on paper!).

___1. If n is odd and the square root of n is a natural number then the square root of n is odd.
___2. For all real numbers a and b, if $a^3 = b^3$ then $a = b$.
___3. The division of 2 rational numbers is rational.
___4. A necessary condition for an integer to be divisible by 8 is that it is divisible by 2.
___5. The square of any odd integer is odd
___6. For all real numbers a and b, if $a^6 = b^6$ then $a = b$.

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8. **(8 points)** `Library/MC/Proofs/ContrapositiveProofs/Parity.pg`

Order 8 of the following sentences so that they form a logical proof by **contrapositive** of the statement:

If the sum of two integers is even then they have the same parity.

- $x+y$ is odd
- Either x is odd and y is even or x is even and y is odd.
- Therefore, $x+y$ is even $\Rightarrow x$ and y have the same parity.
- Without loss of generality assume x is even and y is odd.
- $\exists k, j \in \mathbb{Z}$ such that $x = 2k$ and $y = 2j+1$
- $\exists s$ such that $x+y = 2s+1$
- Assume $x+y$ even implies x and y have the same parity.
- $x+y = 2k+2j+1$
- Suppose x and y are integers with opposite parity.

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9. **(8 points)** `Library/MC/Proofs/ContradictionProofs/Parity.pg`

Order 8 of the following sentences so that they form a logical proof by **contradiction** of the statement:

If the sum of two integers is even then they have the same parity.

- x is even and y is odd or x is odd and y is even
- $x+y = 2k+1+2j = 2(k+j)+1$
- $\exists k, j \in \mathbb{Z}$ such that $x = 2k+1$ and $y = 2j$
- Assume $x+y$ even implies x and y have the same parity.
- Hence x and y have the same parity
- $x+y$ is odd
- Let x and y be opposite parity integers with even sum.
- Presume the provided statement is false.
- Without loss of generality, assume x is odd and y is even
- Let x and y be integers with the same parity but with an odd sum.
- Parity is not knowable without a paring knife

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10. **(8 points)** `Library/MC/Proofs/ContrapositiveProofs/EvenOdd1.pg`

Order 7 of the following sentences so that they form a logical proof by **contrapositive** of the statement:

$$\forall x \varepsilon \mathbb{Z}, 7x+17 \textit{ even} \Rightarrow x \textit{ is odd}$$

- $7x+17 = 2j+1$ for some integer j
- $7x+17 = 2(7k+8)+1$
- Suppose $7x+17$ is not even.
- $\exists k$ such that $x = 2k$
- x is even.
- $7x+17 = 7(2k)+17 = 14k+16+1$
- Suppose x is an integer that is not odd.
- $7x+17$ is odd
- Then a miracle occurred.
- $7x+17$ is even

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11. **(8 points)** `Library/MC/Proofs/ContradictionProofs/NsquaredOddImpliesNodd.pg`

Order 9 of the following sentences so that they form a logical proof by **contradiction** of the statement:

If the square of an integer is odd then the original integer is also odd.

- Let $m \in \mathbb{Z}$ such that m^2 is odd.
- $\exists k \in \mathbb{Z}$ such that $m = 2k+1$.
- So m^2 is odd and m^2 is even.
- m^2 is even
- Suppose m is even.
- Suppose m is an integer such that m^2 is even.
- $\exists k \in \mathbb{Z}$ such that $m = 2k$.
- Presume that the given statement is incorrect.
- Therefore, m is odd
- $\exists j \in \mathbb{Z}$ such that $m^2 = 2j$
- Suppose m is odd.
- $m^2 = 4k^2$

12. **(8 points)** `Library/MC/Proofs/DirectProofs/divisibility1.pg`
Order 5 of the following sentences so that they form a logical proof of the statement:

a divides b and a divides c implies a divides b+c

- $b+c = ka+ka$
- $\exists k$ such that $b = ka$ and $c = ka$
- $\exists k, j$ such that $b = ka$ and $c = ja$
- $b+c = ka+ja$
- a divides $b+c$
- Assume for some integers a, b and c that $b|a$ and $c|a$.
- $b+c = (2k)a$
- Assume $a|b$ and $a|c$ for some integers a, b and c
- $b+c = a(k+j)$