Lecture 1.2: Inclusion-exclusion

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4190, Discrete Mathematical Structures

Combinatorics

Throughout this course, we will be counting things. The mathematical field of counting is called combinatorics.

One of the most basic things to count is the number of elements in a set.

This is easy when the sets are disjoint.

Disjoint (easy). There are 60 cats and 40 dogs at the local shelter. How many animals are there?

Non-disjoint (harder). An apartment complex houses 50 families: 15 own dogs, 20 own cats, and 25 own neither. How many people own both cats and dogs?

Later, we will encounter other counting problems involving permutations, such as:

- How many ways are there to choose 4 teams (out of 25) to play in the College Football Playoff?
- How many ways are there to rank the top 4 teams (out of 25)?

Set partitions

Definition

A partition of set A is a set of one or more nonempty subsets of A: $A_1, A_2, A_3 \ldots$ such that every element of A is in exactly one set. Symbolically,

(i) $A_1 \cup A_2 \cup A_3 \cup \cdots = A$

(ii) If $i \neq j$, then $A_i \cap A_j = \emptyset$.

The subsets A_i are called blocks.

Example

Let $A = \{a, b, c, d\}$. Examples of partitions of A are:

- $\{\{a\}, \{b\}, \{c, d\}\}$
- $\{\{a, b\}, \{c, d\}\}$
- $\{\{a\}, \{b\}, \{c\}, \{d\}\}$

Proposition

If A is a finite set and $\{A_1, \ldots, A_n\}$ is a partition of A, then

$$|A| = |A_1| + |A_2| + \cdots + |A_n| = \sum_{k=1}^{n} |A_k|.$$

Set partitions

Example (easy)

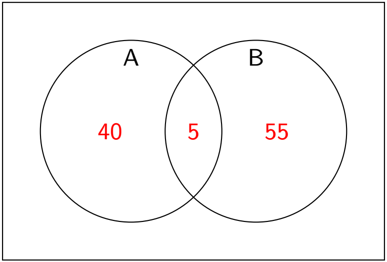
All freshman in the honors college must take one of three classes that are offered at the same time: Math, CS, and Econ.

The enrollments for these classes is 30, 50, and 20. How many honors freshman are there?

Non-disjoint sets

Example (a little harder)

The honors college has 100 students, all of whom major in either Math or CS (or both). If there are 45 math majors and 60 are CS majors, how many double majors are there?



Law of inclusion-exclusion (2 sets)

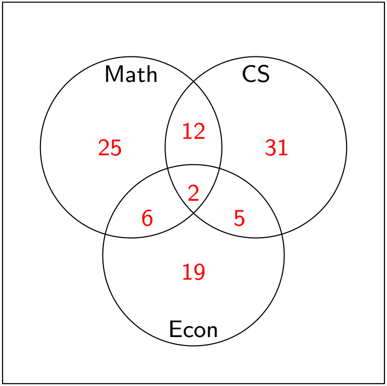
Given finite sets A_1 and A_2,

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Non-disjoint sets

Example (harder)

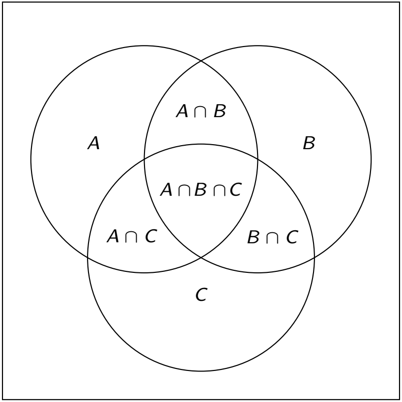
The honors college has 100 students, all of whom major in either Math, CS, or Econ. If there are 45 math majors, 50 CS majors, and 32 econ majors, how many double and triple majors are there?



Inclusion-exclusion (3 sets)

Given finite sets A_1, A_2, A_3,

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| = {} & |A_1| + |A_2| + |A_3| \\ & - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\ & + |A_1 \cap A_2 \cap A_3|.\end{aligned}$$



Probabilities

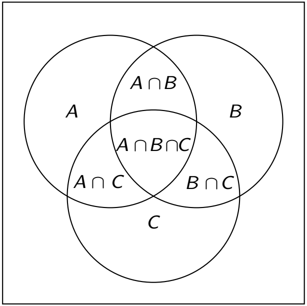
It is straightforward to modify inclusion-exclusion to the theory of probability.

Inclusion-exclusion (3 sets)

Suppose A_1, A_2, A_3 are events. Then

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) \\ &\quad - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\ &\quad + P(A_1 \cap A_2 \cap A_3). \end{aligned}$$

Here, $A \cup B$ means A or B, and $A \cap B$ means A and B.



Remarks

- Think of probabilities as areas of regions.
- The total area of the universe must be 1.
- A, B are independent if $P(A \cap B) = P(A)P(B)$.
- Bayes' theorem for conditional probability says:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ if } P(B) \neq 0.$$