Lecture 2.2: Tautology and contradiction

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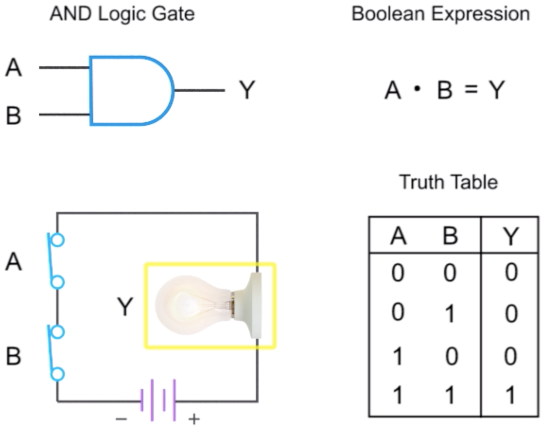
Math 4190, Discrete Mathematical Structures

Motivation

Digital electronic circuits are made from large collections of logic gates, which are physical devices that implement Boolean functions.



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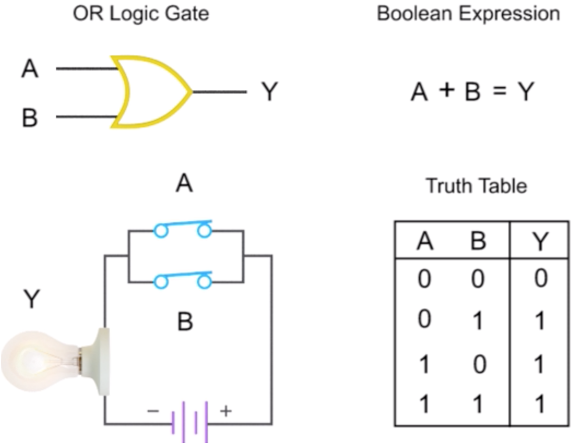

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1

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Motivation

Understanding digital circuits requires an understanding of Boolean logic.

Recall that we have seen the following logical operations:

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$q \rightarrow p$	$\neg q \rightarrow \neg p$	$p \leftrightarrow q$
0	0	0	0	1	1	1	1
0	1	0	1	1	0	1	0
1	0	0	1	0	1	0	0
1	1	1	1	1	1	1	1

Note that:

- $p \rightarrow q$ has the same truth table as $(\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q)$, or just $\neg p \vee q$.
- $q \rightarrow p$ has the same truth table as $(\neg p \wedge \neg q) \vee (p \wedge \neg q) \vee (p \wedge q)$, or just $p \vee \neg q$.
- $p \leftrightarrow q$ has the same truth table as $(\neg p \wedge \neg q) \vee (p \wedge q)$.

Not surprisingly, *every* Boolean function can be written with $\wedge$, $\vee$, and $\neg$.

Even with just these operations, many propositions are the same. For example, $\neg(p \wedge q)$ and $\neg p \vee \neg q$ have the same meaning.

Compound propositions

If p, q, and r are propositions, we say that the compound proposition

$$c = (p \wedge q) \vee (\neg q \wedge r)$$

is generated by p, q, and r.

The value of c is determined by the $2^3 = 8$ possibile combinations of truth values for p, q, and r. We can describe this via a truth table:

p	q	r	$p \wedge q$	$\neg q$	$\neg q \wedge r$	$(p \wedge q) \vee (\neg q \wedge r)$
0	0	0	0	1	0	0
0	0	1	0	1	1	1
0	1	0	0	0	0	0
0	1	1	0	0	0	0
1	0	0	0	1	0	0
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	0	0	1

Note that the first three colums are the numbers $0, \dots, 7$ in binary.

In general, if c is generated by n propositions, then its truth table will have 2^n rows.

Compound propositions

Let S be any set of propositions. A proposition generated by S is any valid combination of propositions in S with conjunction, disjunction, and negation.

Definition (formal)

(a) If $p \in S$, then p is a proposition generated by S, and

(b) If x and y are propositions generated by S, then so are (x), $\neg x$, $x \vee y$, and $x \wedge y$.

There is a standard "order of operations":

1. Negation: $\neg$
2. Conjunction: $\wedge$
3. Disjunction: $\vee$
4. Conditional operation: $\rightarrow$
5. Biconditional operation: $\leftrightarrow$

Despite this, we will avoid writing potentially ambiguous statements like $p \vee q \wedge r$. Expressions like the following should be unambiguous:

(a) $p \wedge q \wedge r$ is $(p \wedge q) \wedge r$

(b) $\neg p \vee \neg r$ is $(\neg p) \vee (\neg r)$

(c) $\neg\neg p$ is $\neg(\neg p)$.

Tautologies

Definition

An expression involving logical variables that is true in all cases is a tautology. We use the number 1 to symbolize a tautology.

Examples

The following are all tautologies:

(a) $(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$

(b) $p \vee \neg p$

(c) $(p \wedge q) \rightarrow p$

(d) $q \rightarrow (p \vee q)$

(e) $(p \vee q) \leftrightarrow (q \vee p)$

Contradictions

Definition

An expression involving logical variables that is false in all cases is a contradiction. We use the number 0 to symbolize a contradiction.

Examples

The following are contradictions:

(a) $p \wedge \neg p$

(b) $(p \vee q) \wedge (\neg p) \wedge (\neg q)$