Lecture 3.1: The pigeonhole principle

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A "very obvious" fact

Pigeonhole principle

If there are $n+1$ pigeons in n holes, then some hole contains at least 2 pigeons.

In this lecture, we'll see some rather surprising consequences of this seemingly simple fact.

Applications of the pigeonhole principle

Groups of people

- Among 13 people, there are two with birthdays in the same month.
- In NYC, there are two non-bald people with the same number of hairs on their head.
- In a class of n people, there are two with the same number of friends in the class.

An application to graph theory

Claim

Every graph contains two nodes with the same degree.

A geometric application

Claim

If you draw 5 points on a sphere, then it is always possible to cut it so that some hemisphere contains at least 4 of them (possibly on the boundary, i.e., on the cut).

An application to number theory

Let S be a set of distinct 101 integers chosen from $1, 2, \ldots, 200$.

Note that there are $\binom{200}{101} \approx 8.97 \times 10^{58}$ possibilities for S.

Claims

(i) S contains two consecutive integers.

(ii) S contains two integers such that one of them is divisible by the other.

Generalizations

A stronger pigeonhole principle

Let n and r be integers. If $n(r-1)+1$ objects are placed in n boxes, then some box contains at least r objects.

Examples

- In any group of $12(3-1)+1=25$ people, at least three were born in the same month.
- In a school of $366(67-1)+1=24156$ students, at least 67 share a birthday.

The strong pigeonhole principle

Let $q_1,\dots,q_n \in \mathbb{Z}^+$. If $q_1+\cdots+q_n-n+1$ objects are put into n boxes, then either the 1^{st} box contains $\geq q_1$ objects, or the 2^{nd} contains $\geq q_2$ objects, ..., or the n^{th} box contains $\geq q_n$ objects.

Note that the special case of $q_1=\cdots q_n=1$ is the pigeonhole priniple.

The special case of $q_1=\cdots q_n=r$ yields the "stronger pigeonhole principle," above.