

## TOPICS: LINEAR MAPS, INNER PRODUCTS, AND ORTHOGONALITY

1. Let  $T: V \rightarrow W$  be a linear map between vector spaces. Prove that  $\ker(T) := \{v \in V \mid T(v) = 0\}$  is a subspace of  $V$ .
2. Let  $V = \mathcal{C}^1(\mathbb{R})$ , the vector space of differentiable real-valued functions. Consider the linear operator  $T = \frac{d}{dt} + 3$ .
  - (a) The kernel of  $T$  can be characterized precisely by the set of all functions that solve a particular differential equation. Write down this equation.
  - (b) Find the general solution for the differential equation you found in Part (a), and hence an explicit formula for  $\ker(T)$ .
  - (c) Write down an explicit basis for the solution space,  $\ker(T)$ . What is the dimension of this vector space?
3. Let  $\mathbf{v} = (3, 4) \in \mathbb{R}^2$ .
  - (a) Compute  $\|\mathbf{v}\| := \sqrt{\mathbf{v} \cdot \mathbf{v}}$ .
  - (b) Recall that  $\{\mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1)\}$  is an *orthonormal basis* for  $\mathbb{R}^2$ . Decompose  $\mathbf{v}$  into this basis, i.e., write  $\mathbf{v} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$  for some  $a_1, a_2 \in \mathbb{R}$ .
  - (c) Sketch  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{v}$  in  $\mathbb{R}^2$ . Graphically show what  $a_1$  and  $a_2$  represent in terms of the projection of  $\mathbf{v}$  onto the unit vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$ .
  - (d) The set  $\{\mathbf{v}_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), \mathbf{v}_2 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\}$  is also an orthonormal basis. Decompose  $\mathbf{v}$  into this basis, i.e., write  $\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$  for some  $b_1, b_2 \in \mathbb{R}$ .
  - (e) On a new set of axes, sketch  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}$  in  $\mathbb{R}^2$ . Graphically show what  $b_1$  and  $b_2$  represent in terms of the projection of  $\mathbf{v}$  onto the unit vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .
4. For this problem, consider the vector space  $V = \mathbb{R}^3$  and use the vector dot product as the inner product.
  - (a) Show that the set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ , where
 
$$\mathbf{v}_1 = (1, 2, -2), \quad \mathbf{v}_2 = (0, 1, 1), \quad \mathbf{v}_3 = (-4, 1, -1).$$
 is an orthogonal set, but not orthonormal.
  - (b) Normalize  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$  to get an *orthonormal* basis of  $\mathbb{R}^3$ . That is, compute the following:
 
$$\mathcal{B} = \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\} \quad \text{where} \quad \mathbf{n}_i = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}.$$
  - (c) Use the dot product to express the vector  $\mathbf{w} = (1, 2, 3)$  in terms of  $\mathbf{n}_1$ ,  $\mathbf{n}_2$ , and  $\mathbf{n}_3$ . That is, find  $C_1$ ,  $C_2$ , and  $C_3$  such that

$$\mathbf{w} = C_1\mathbf{n}_1 + C_2\mathbf{n}_2 + C_3\mathbf{n}_3.$$

5. Let  $\mathbb{R}_3[x] = \{a_3x^3 + a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R}\}$ , the vector space of polynomials of degree at most 3. Define the following *inner product* on  $\mathbb{R}_3[x]$ :

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx.$$

- (a) Verify that this is indeed an inner product on  $\mathbb{R}_3[x]$ .  
(b) Consider the two sets

$$\mathcal{B}_1 = \{1, x, 3x^2 - 1, 5x^3 - 3x\}, \quad \mathcal{B}_2 = \{1, x, x^2, x^3\}$$

that are both bases for  $\mathbb{R}_3[x]$ . Show that  $\mathcal{B}_1$  is an orthogonal set, but  $\mathcal{B}_2$  is not. (The set  $\mathcal{B}_1$  are the first four *Legendre polynomials*,  $P_n(x)$  for  $n = 0, \dots, 3$ . When we study Sturm-Liouville theory, we will see why the Legendre polynomials are always orthogonal!)

- (c) For each  $f \in \mathcal{B}_1$ , compute the *norm* of  $f$ , which is defined as  $\|f\| = \langle f, f \rangle^{1/2}$ . Find an *orthonormal* basis for  $\mathbb{R}_3[x]$  by normalizing the elements in  $\mathcal{B}_1$ .  
(d) Consider the polynomial  $f(x) = 3x^3 - 2x^2 + 4$ . Use orthogonality to write  $f(x)$  using the elements in  $\mathcal{B}_1$ . That is, find  $C_0, C_1, C_2$ , and  $C_3$  such that

$$3x^3 - 2x^2 + 4 = C_0 + C_1x + C_2(3x^2 - 1) + C_3(5x^3 - 3x).$$