Lecture 2.3: Inhomogeneous differential equations and affine spaces

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The main idea

We've seen how to solve (some) linear homogeneous ODEs. The set of solutions form a vector space.

In this lecture, we'll look at linear inhomogeneous equations.

Definition (1st order)

Consider $y' + a(t)y = g(t)$, where $g(t) \neq 0$.

Define the related homogeneous equation to be $y_h' + a(t)y_h = 0$, and say its general solution is $y_h(t) = C_1 y_1(t)$.

Theorem

The general solution to $y' + a(t)y = g(t)$ has the form

$$y(t) = y_h(t) + y_p(t) = \Big\{ C_1 y_1(t) + y_p(t) \mid C_1 \in \mathbb{C} \Big\},$$

where $y_p(t)$ is *any particular solution* to the original (inhomogeneous) ODE.

Fundamental theorem

Theorem

There general solution to $y' + a(t)y = g(t)$ has the form

$$y(t) = y_h(t) + y_p(t) = \Big\{ C_1 y_1(t) + y_p(t) \mid C_1 \in \mathbb{C} \Big\},$$

where $y_p(t)$ is *any particular solution* to the original ODE.

Proof

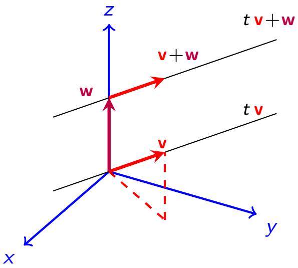
We'll show that $y(t) - y_p(t)$ solves the homogeneous equation, $y_h' + a(t)y_h = 0$.

Similar problems different areas of mathematics

1. Parametrize a line in $\mathbb{R}^n$.
2. Parametrize a plane in $\mathbb{R}^n$.
3. Solve the underdetermined system $\mathbf{Ax} = \mathbf{b}$.
4. Solve the differential equation $y' + 4y = 8$.
5. Solve the differential equation $y'' + 4y = 8$.

Parametrize a line in $\mathbb{R}^n$

Suppose we want to write the equation for a line that contains a vector $\mathbf{v} \in \mathbb{R}^n$:



This line, which *contains the zero vector*, is $t\mathbf{v} = \{t\mathbf{v} : t \in \mathbb{R}\}$.

Now, what if we want to write the equation for a line parallel to $\mathbf{v}$?

This line, which *does not contain the zero vector*, is

$$t\mathbf{v} + \mathbf{w} = \{t\mathbf{v} + \mathbf{w} : t \in \mathbb{R}\}.$$

Note that ANY particular $\mathbf{w}$ on the line will work!!!

Solve an underdetermined system $\mathbf{Ax} = \mathbf{b}$

Suppose we have a system of equations that has "too many variables," so there are infinitely many solutions.

For example:

$$\begin{aligned} 2x + y + 3z &= 4 \\ 3x - 5y - 2z &= 6 \end{aligned} \qquad \text{"}\mathbf{Ax} = \mathbf{b} \text{ form"}: \qquad \begin{bmatrix} 2 & 1 & 3 \\ 3 & -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

How to solve:

1. Solve the related homogeneous equation $\mathbf{Ax} = \mathbf{0}$ (this is $\ker(\mathbf{A})$, i.e., the "nullspace");
2. Find any particular solution $\mathbf{x}_p$ to $\mathbf{Ax} = \mathbf{b}$;
3. Add these together to get the general solution: $\mathbf{x} = \ker(\mathbf{A}) + \mathbf{x}_p$.

This works because geometrically, the solution space is just a line, plane, etc.

Here are two possible ways to write the solution:

$$\mathbf{x} = C \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x} = C \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 10 \\ 8 \\ -8 \end{bmatrix}.$$

Linear differential equations

Example

Solve the differential equation $y' + 4y = 8$.

Steps:

1. Solve the related homogeneous equation $y_h' + 4y_h = 0$. The solution is $y_h(t) = Ce^{-4t}$.
2. Find any particular solution $y_p(t)$ to $y' + 4y = 8$. By inspection, we see that $y_p(t) = 2$ works.
3. Add these together to get the general solution:

$$y(t) = y_h(t) + y_p(t) = Ce^{-4t} + 2.$$

Note that while the general solution above is unique, its presentation need not be.

For example, we could write it this way:

$$y(t) = y_h(t) + y_p(t) = Ce^{-4t} + (5e^{-4t} + 2).$$

Linear differential equations

Example

Solve the differential equation $y'' + 4y = 8$.

Steps:

1. Solve the related homogeneous equation $y_h'' + 4y_h = 0$. The solution is $y_h(t) = C_1 \cos 2t + C_2 \sin 2t$.
2. Find any particular solution $y_p(t)$ to $y'' + 4y = 8$. By inspection, we see that $y_p(t) = 2$ works.
3. Add these together to get the general solution:

$$y(t) = y_h(t) + y_p(t) = C_1 \cos 2t + C_2 \sin 2t + 2.$$

Note that while the general solution above is unique, its presentation need not be.

For example, we could write it this way:

$$y(t) = y_h(t) + y_p(t) = C_1 \cos 2t + C_2 \sin 2t + (5 \cos 2t - 3 \sin 2t + 2).$$

Affine spaces

The solutions to linear homogeneous ODEs form a vector space.

First order: $\Big\{C_1 y_1(t) : C_1 \in \mathbb{R}\Big\}$.

Second order: $\Big\{C_1 y_1(t) + C_2 y_2(t) : C_1, C_2 \in \mathbb{R}\Big\}$.

The solutions to linear inhomogeneous ODEs have the form:

$$\Big\{C_1 y_1(t) + y_p(t) : C_1 \in \mathbb{R}\Big\} \qquad \text{or} \qquad \Big\{C_1 y_1(t) + C_2 y_2(t) + y_p(t) : C_1, C_2 \in \mathbb{R}\Big\}.$$

These are *not* vector spaces, but they are "close". They are called affine spaces.

Intuitively

An affine space "looks like" a line, plane, etc., but *not* through the origin.

Definition

An affine space is a set A (of vectors) and a set $\mathbb{F}$ (of scalars) such that for some particular vector $w \in A$, the set

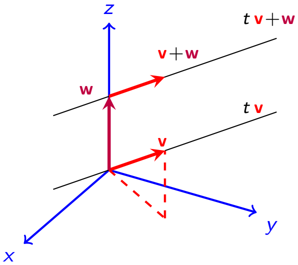
$$A - w := \big\{v - w : v \in A\big\}$$

is a vector space over $\mathbb{F}$.

A 1D geometric example

Take any nonzero vector $\mathbf{v} \in \mathbb{R}^3$. The line L containing it is a vector space:

$$L = t\mathbf{v} = \{t\mathbf{v} : t \in \mathbb{R}\}.$$



Any parallel line A (not through $\mathbf{0}$) is an affine space.

Recall that an equation for such as line is

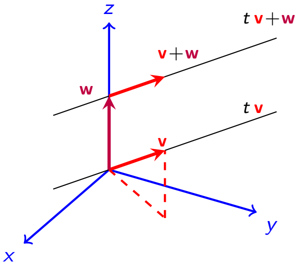
$$A = t\mathbf{v} + \mathbf{w} = \{t\mathbf{v} + \mathbf{w} : t \in \mathbb{R}\}.$$

where ANY particular $\mathbf{w}$ on the line will work!!!

This line satisfies the definition of an affine space because if we subtract $\mathbf{w}$ from it, we get a vector space:

$$A - \mathbf{w} = \{(t\mathbf{v} + \mathbf{w}) - \mathbf{w} : t \in \mathbb{R}\} = \{t\mathbf{v} : t \in \mathbb{R}\} = L.$$

A 1st order ODE example



Suppose $y_1(t) \neq 0$ solves $y_h' + a(t)y_h = 0$.

The solution space $L = \{C_1 y_1(t) \mid C_1 \in \mathbb{R}\}$ is a vector space.

Now, suppose $y_p(t)$ solves $y' + a(t)y = g(t)$, where $g(t) \neq 0$. The set of solutions

$$A := \Big\{ C_1 y_1(t) + y_p(t) \mid C_1 \in \mathbb{R} \Big\}$$

is ***not*** a vector space. But it is an affine space because the set

$$A - y_p(t) = \Big\{ [C_1 y_1(t) + y_p(t)] - y_p(t) \mid C_1 \in \mathbb{R} \Big\} = \Big\{ C_1 y_1(t) \mid C_1 \in \mathbb{R} \Big\} = L$$

is a vector space.

Change of variables

Remark

When we do a change of variables,

e.g., let $(u_1, u_2) = (x_1 - a, x_2 - b)$ in $\mathbb{R}^2$

or, let $u(t) = y(t) - y_p(t)$,

all we're doing is making an inhomogeneous equation into a homogeneous one.

Example

The rate of change of the temperature of a cup of coffee is proportional to the *difference* between the coffee's temperature and the ambient temperature:

$$y' = -k(y - 72).$$