Lecture 5.3: The transport and wave equations

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Motivation

Some common one-dimensional PDEs

We've seen the heat equation: $u_t = c^2 u_{xx}$. In this lecture, we will introduce the transport equation, from which we will derive the wave equation: $u_{tt} = c^2 u_{xx}$.

Transport left

Example 1

Consider the following PDE involving a function $u(x, t)$:

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0.$$

Transport right

Example 2

Consider the following PDE involving a function $u(x, t)$:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

The wave equation

Example 3

Consider the following PDE involving a function $u(x,t)$:

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)u=\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0$$

The three most common two-variable PDEs

Summary

Let $u(x, t)$ be a function of position x and time t. Then

- the heat equation is $u_t = c^2 u_{xx}$,
- the wave equation is $u_{tt} = c^2 u_{xx}$.

One more

Let $u(x, y)$ be a function of position (x, y). Then

- Laplace's equation is $u_{xx} + u_{yy} = 0$.

Example 3

Solve the following B/IVP for the wave equation:

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, \qquad u(x,0) = x(L-x), \quad u_t(x,0) = 1\,.$$

Example 3 (cont.)

The general solution to the following BVP for the wave equation:

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0, \qquad u(x,0) = x(L-x), \quad u_t(x,0) = 1\,.$$

is $u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{cn\pi t}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$. Now, we'll solve the remaining IVP.