Read Chapters 8.4–5 of *Visual Group Theory*, Chapter 9.2 of *IBL Abstract Algebra*, or Chapters 10.3 and 12.1 of *AATA*. Then write up solutions to the following exercises.

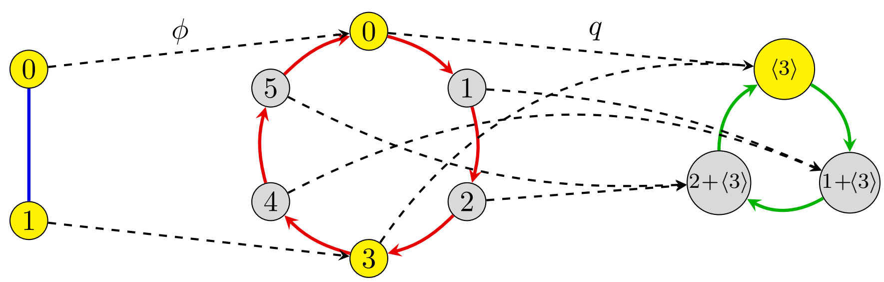
1. Let $(\mathbb{Q}, +)$ be the group of rational numbers under addtion, $(\mathbb{Q}^*, \cdot\,)$ the group of non-zero rational numbers under multiplication, and $(\mathbb{Q}^+, \cdot\,)$ the group of positive rational numbers under multiplication.

   (a) Show that $(\mathbb{Q}^*, \cdot\,) \cong (\mathbb{Q}^+, \cdot\,) \times C_2$. [*Hint*: Recall that $C_2 = \{e^{0\pi i}, e^{\pi i}\} = \{1, -1\}$.]

   (b) Describe the quotient groups $(\mathbb{Q}, +)/\langle -1 \rangle$ and $(\mathbb{Q}^*, \cdot\,)/\langle -1 \rangle$. In particular, what do the elements (cosets) look like?

   (c) Use the Fundamental Homomorphism Theorem to prove that $(\mathbb{Q}^*, \cdot\,)/\langle -1 \rangle \cong (\mathbb{Q}^+, \cdot\,)$.

2. For Parts (a)–(d), a group G is given together with a normal subgroup H. Illustrate the embedding $\phi\colon H \to G$, and the quotient map $q\colon G \to G/H$, chained together so that $\mathrm{im}(\phi) = \ker(q)$. An example for $G = \mathbb{Z}_6$ and $H = \mathbb{Z}_2$ is shown below:



$H = \mathbb{Z}_2$ $\qquad$ $H \cong \mathrm{im}(\phi) = \langle 3 \rangle < G$ $\qquad$ $G/\langle 3 \rangle \cong \mathbb{Z}_3$

   (a) $G = \mathbb{Z}_6$, $H = \mathbb{Z}_3$,

   (b) $G = D_3$, $H = C_3$,

   (c) $G = A_4$, $H = V_4$,

   (d) $G = S_n$, $H = A_n$ [don't draw the actual Cayley graphs for this one, just the maps].

   Now, answer each of the following questions about each of your answers to Parts (a)–(d).

   (e) What map θ into H would satisfy the equation $\mathrm{im}\,\theta = \ker\phi$? Choose one with the smallest possible domain.

   (f) What map θ' from G/H would satisfy the equation $\mathrm{im}(q) = \ker(\theta')$? Choose one with the smallest possible codomain.

   (g) Add the two maps θ and θ' to your illustration.

   (h) The new chain of four homomorphisms is called a *short exact sequence*. It is one way to use homomorphisms to illustrate quotients, and it shows a connection betwen embeddings and quotient maps. Given a normal subgroup $H \trianglelefteq G$, show how to create a short exact sequence involving G and H.

3. Let $A \leq B$ and $B \trianglelefteq G$. In this problem, you will prove the *Diamond Isomorphism Theorem.*

   (a) Show that $AB := \{ab : a \in A,\, b \in B\}$ and $BA := \{ba : a \in A,\, b \in B\}$ are equal as sets.

   (b) Show that AB is a subgroup of G.

   (c) Show that $B \trianglelefteq AB$ and $A \cap B \trianglelefteq A$.

   (d) Show that $A/(A \cap B) \cong AB/B$. [*Hint:* Construct a homomorphism $\phi\colon A \to AB/B$ that has kernel $A \cap B$, then apply the FHT.]

   (e) Draw a diagram, or lattice, of G and its subgroups AB, A, B, and $A \cap B$. Interpret the result in Part (c) in terms of this diagram.

4. For each part below, consider the group $G = \langle A, B\rangle$ generated by the two matrices given. Assume that matrix multiplication is the binary operation, and $i = \sqrt{-1}$. To what common group is G isomorphic? Write down an explicit isomorphism (you only need to define it for the generators), and a group presentation for G.

   (a) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

   (b) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$

   (c) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$

5. In this exercise, you will prove that if A and B are normal subgroups and $AB = G$, then

$$G/(A \cap B) \cong (G/A) \times (G/B).$$

   (a) Consider the following map:

$$\phi\colon AB \longrightarrow (G/A) \times (G/B)\,, \qquad \phi(g) = (gA, gB).$$

   Show that ϕ is a homomorphism.

   (b) Show that ϕ is surjective. That is, given any (g_1A, g_2B), show that there is some $g = ab \in AB$ such that $\phi(g) = (g_1A, g_2B)$. [*Hint*: Try $g = a_2b_1$.]

   (c) Find $\ker(\phi)$ [you need to prove your answer is correct] and then apply the Fundamental Homomorphism Theorem.

6. For the numbers below, list all abelian groups of that order by writing each one as a product of cyclic groups of prime power order. Then, determine which group it is isomorphic to of the form $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, where $n_{i+1} \mid n_i$.

   (a) 16

   (b) 54

   (c) 400

   (d) p^2q, where p and q are distinct primes.