Read Chapters 15.1–15.3 of *AATA*. Then write up solutions to the following exercises.

1. For each of the following rings $R$, determine the zero divisors (right and left, if appropriate), and the set $U(R)$ of units.

   (a) The set $\mathcal{C}^1$ of continuous real-valued functions $f\colon \mathbb{R} \to \mathbb{R}$.

   (b) The polynomial ring $\mathbb{R}[x]$.

   (c) $\mathbb{Z} \times \mathbb{Z}$, where addition and multiplication are defined componentwise.

   (d) $\mathbb{R} \times \mathbb{R}$, where addition and multiplication are defined componentwise.

2. Prove that if a left ideal $I$ of a ring $R$ contains a unit, then $I = R$.

3. Let $I$ and $J$ be ideals of a ring $R$.

   (a) Prove that $I + J$, $I \cap J$, and $IJ$ are ideals of $R$.

   (b) If $R$ is commutative, then the set

   $$(I : J) = \{r \in R \mid rJ \subseteq I\}$$

   is called the *ideal quotient* or *colon ideal* of $I$ and $J$. Show that $(I : J)$ is an ideal of $R$.

   (c) Consider the ideals $I = 4\mathbb{Z}$ and $J = 6\mathbb{Z}$ of the ring $R = \mathbb{Z}$. Compute $I + J$, $I \cap J$, $IJ$, $(I : J)$, and $(J : I)$.

   (d) Repeat Part (c) for the ideals $I = m\mathbb{Z}$ and $J = n\mathbb{Z}$ of $R = \mathbb{Z}$.

4. The left ideal generated by $X \subseteq R$ is defined as

   $$(X) := \bigcap \{I : \ I \text{ is a left ideal s.t. } X \subseteq I \subseteq R\}.$$

   (a) Prove that the left ideal generated by $X$ is

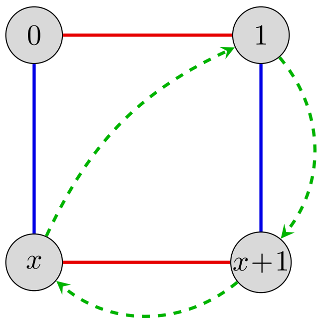
   $$(X) = \{r_1x_1 + \cdots + r_nx_n : n \in \mathbb{N},\ r_i \in R,\ x_i \in X\}.$$

   (b) The two-sided ideal generated by $X \subseteq R$ is defined by relacing "left" with "two-sided" in the definition above. Prove this this is also equal to

   $$\{r_1x_1s_1 + \cdots + r_nx_ns_n : n \in \mathbb{N},\ r_i, s_i \in R,\ x_i \in X\}.$$

   (c) Find a (non-commutive) ring $R$ and a set $X$ such that the left and two-sided ideals generated by $X$ are different.

5. The finite field $\mathbb{F}_4$ on 4 elements can be constructed as the quotient of the polynomial $\mathbb{Z}_2[x]$ by the ideal $I = (x^2 + x + 1)$ generated by the irreducible polynomial $x^2 + x + 1$. The figure below shows a Cayley diagram, and multiplication and addition tables for the finite field $\mathbb{Z}_2[x]/(x^2 + x + 1) \cong \mathbb{F}_4$.




| $+$ | $0$ | $1$ | $x$ | $x+1$ |
|---|---|---|---|---|
| $0$ | $0$ | $1$ | $x$ | $x+1$ |
| $1$ | $1$ | $0$ | $x+1$ | $x$ |
| $x$ | $x$ | $x+1$ | $0$ | $1$ |
| $x+1$ | $x+1$ | $x$ | $1$ | $0$ |

| $\times$ | $1$ | $x$ | $x+1$ |
|---|---|---|---|
| $1$ | $1$ | $x$ | $x+1$ |
| $x$ | $x$ | $x+1$ | $1$ |
| $x+1$ | $x+1$ | $1$ | $x$ |

(a) Find a degree-3 polynomial $f \in \mathbb{Z}_2[x]$ that is irreducible over $\mathbb{Z}_2$, and a degree-2 polynomial $g \in \mathbb{Z}_3[x]$ that is irreducible over $\mathbb{Z}_3$. [*Hint*: Any polynomial with no roots in the "prime field" $\mathbb{Z}_p$ will work.]

(b) Construct Cayley diagrams, addition, and multiplication tables for the finite fields

$$\mathbb{F}_8 \cong \mathbb{Z}_2[x]/(f) \qquad \text{and} \qquad \mathbb{F}_9 \cong \mathbb{Z}_3[x]/(g)\,.$$

6. Prove the Fundamental Homomorphism Theorem (FHT) for rings: If $\phi\colon R \to S$ is a ring homomorphism, then $\operatorname{Ker}\phi$ is a two-sided ideal of $R$, and $R/\operatorname{Ker}\phi \cong \operatorname{Im}\phi$. You may assume the FHT for groups.