1. Let $p \in \mathbb{N}$ be a fixed prime. For each of the three ideals $I = (p)$, $(x)$, and $(x,p)$ in the ring $R = \mathbb{Z}[x]$, do the following steps:

   (i) Describe the elements of the ideal formally, as $I = \{\quad : \quad\}$.

   (ii) Characterize the polynomials in $I$ in plain English.

   (iii) Determine whether $I$ is maximal and/or prime.

   (iv) Describe the quotient ring $R/I$.

   Then, repeat the above steps for these ideals but in the ring $\mathbb{Q}[x]$.

2. Let $R$ be a commutative ring with 1.

   (a) Prove that $R$ is an integral domain if and only if 0 is a prime ideal.

   (b) Prove that an ideal $P \subseteq R$ is prime if and only if $R/P$ is an integral domain.

   (c) Show that every maximal ideal is prime.

   (d) Find the group of units $U(R)$ and the maximal ideal(s) of the ring

   $$R = \left\{ \frac{a}{b} : a, b \in \mathbb{Z},\ \gcd(a,b) = 1,\ p \nmid b \right\},$$

   where $p$ is a fixed prime number.

3. Let $R$ be a principal ideal domain (PID). A *common multiple* of $a, b \in R^*$ is an element $m$ such that $a \mid m$ and $b \mid m$. Moreover, $m$ is a *least common multiple* (LCM) if $m \mid n$ for any other common multiple $n$ of $a$ and $b$.

   (a) Prove that any $a, b \in R^*$ have an LCM.

   (b) Prove that an LCM of $a$ and $b$ is unique up to multiplication of associates, and can be characterized as a generator of the (principal) ideal $I := (a) \cap (b)$.

4. For any $x = r + s\sqrt{m} \in \mathbb{Q}(\sqrt{m})$, define the *norm* of $x$ to be $N(x) = r^2 - ms^2$.

   (a) Show that $N(xy) = N(x)N(y)$.

   (b) Show that $N(x) \in \mathbb{Z}$ if $x \in R_m$.

   (c) Show that $u \in U(R_m)$ if and only if $|N(u)| = 1$.

   (d) Show that $U(R_{-1}) = \{\pm 1, \pm i\}$, $U(R_{-3}) = \{\pm 1, \pm(1 \pm \sqrt{3})/2\}$, and $U(R_m) = \{\pm 1\}$ for all other negative square-free $m \in \mathbb{Z}$.