Lecture 3.3: Normal subgroups

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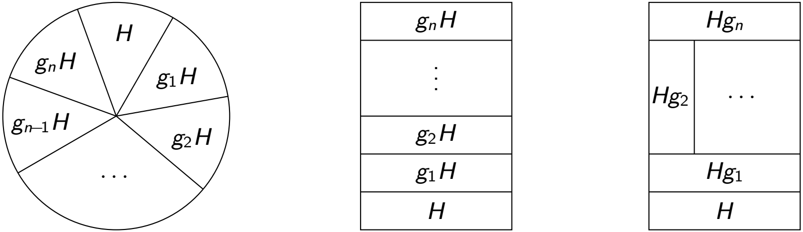
Math 4120, Modern Algebra

Overview

Last time, we learned that for any subgroup $H \leq G$:

- the left cosets of H partition G;
- the right cosets of H partition G;
- these partitions need not be the same.

Here are some visualizations of this idea:



Subgroups whose left and right cosets agree have very special properties, and this is the topic of this lecture.

Normal subgroups

Definition

A subgroup H of G is a normal subgroup of G if $gH = Hg$ for *all* $g \in G$. We denote this as $H \lhd G$, or $H \trianglelefteq G$.

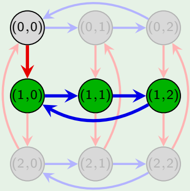
Observation

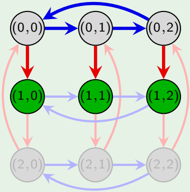
Subgroups of abelian groups are always normal, because for any $H < G$,

$$aH = \{ah \colon h \in H\} = \{ha \colon h \in H\} = Ha\,.$$

Example

Consider the subgroup $H = \langle (0,1) \rangle = \{(0,0), (0,1), (0,2)\}$ in the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ and take $g = (1,0)$. Addition is done modulo 3, componentwise. The following depicts the equality $g + H = H + g$:





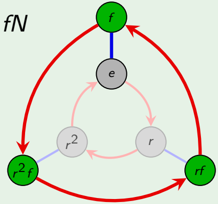
Normal subgroups of nonabelian groups

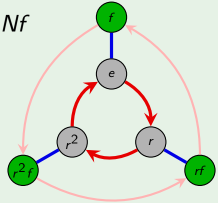
Since subgroups of abelian groups are always normal, we will be particularly interested in normal subgroups of non-abelian groups.

Example

Consider the subgroup $N = \{e, r, r^2\} \leq D_3$.

The cosets (left or right) of N are $N = \{e, r, r^2\}$ and $Nf = \{f, rf, r^2f\} = fN$. The following depicts this equality; the coset $fN = Nf$ are the green nodes.





Normal subgroups of nonabelian groups

Here is another way to visualze the normality of the subgroup, $N = \langle r \rangle \leq D_3$:

fN	f	rf	$r^2 f$
N	e	r	r^2

Nf	f	rf	$r^2 f$
N	e	r	r^2

On contrast, the subgroup $H = \langle f \rangle \leq D_3$ is not normal:

$r^2 H$	$r^2 f$	r^2
rH	r	rf
H	e	f

	Hr	Hr^2	
Hr	$r^2 f$	r^2	Hr^2
Hr	r	rf	Hr^2
H	e	f	

Proposition

If H is a subgroup of G of index $[G : H] = 2$, then $H \lhd G$.

Conjugate subgroups

For a fixed element $g \in G$, the set

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

is called the conjugate of H by g.

Observation 1

For any $g \in G$, the conjugate gHg^{-1} is a subgroup of G.

Proof

1. Identity: $e = geg^{-1}$. ✓
2. Closure: $(gh_1g^{-1})(gh_2g^{-1}) = gh_1h_2g^{-1}$. ✓
3. Inverses: $(ghg^{-1})^{-1} = gh^{-1}g^{-1}$. ✓ □

Observation 2

$gh_1g^{-1} = gh_2g^{-1}$ if and only if $h_1 = h_2$. □

On the homework, you will show that H and gHg^{-1} are isomorphic subgroups. (Though we don't yet know how to do this, or precisely what it means.)

How to check if a subgroup is normal

If $gH = Hg$, then right-multiplying both sides by g^{-1} yields $gHg^{-1} = H$.

This gives us a new way to check whether a subgroup H is normal in G.

Useful remark

The following conditions are all equivalent to a subgroup $H \leq G$ being normal:

(i) $gH = Hg$ for all $g \in G$; ("left cosets are right cosets");

(ii) $gHg^{-1} = H$ for all $g \in G$; ("only one conjugate subgroup")

(iii) $ghg^{-1} \in H$ for all $g \in G$; ("closed under conjugation").

Sometimes, one of these methods is *much* easier than the others!

For example, all it takes to show that H is not normal is finding *one element* $h \in H$ for which $ghg^{-1} \notin H$ for some $g \in G$.

As another example, if we happen to know that G has a unique subgroup of size $|H|$, then H *must* be normal. (Why?)