Lecture 4.3: The fundamental homomorphism theorem

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

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Motivating example (from the previous lecture)

Define the homomorphism \( \phi : Q_4 \to V_4 \) via \( \phi(i) = v \) and \( \phi(j) = h \). Since \( Q_4 = \langle i, j \rangle \):

\[
\begin{align*}
\phi(1) &= e , & \phi(-1) &= \phi(i^2) = \phi(i)^2 = v^2 = e , \\
\phi(k) &= \phi(ij) = \phi(i)\phi(j) = vh = r , & \phi(-k) &= \phi(ji) = \phi(j)\phi(i) = hv = r , \\
\phi(-i) &= \phi(-1)\phi(i) = ev = v , & \phi(-j) &= \phi(-1)\phi(j) = eh = h .
\end{align*}
\]

Let’s quotient out by \( \text{Ker } \phi = \{-1, 1\} \):

\[
\begin{align*}
Q_4 \text{ organized by the } \text{subgroup } K = \langle -1 \rangle \\
\text{left cosets of } K \text{ are near each other} \\
\text{collapse cosets into single nodes}
\end{align*}
\]

Key observation

\( Q_4 / \text{Ker}(\phi) \cong \text{Im}(\phi) \).
The Fundamental Homomorphism Theorem

The following result is one of the central results in group theory.

**Fundamental homomorphism theorem (FHT)**

If $\phi: G \to H$ is a homomorphism, then $\text{Im}(\phi) \cong G/\text{Ker}(\phi)$.

The FHT says that every homomorphism can be decomposed into two steps: (i) quotient out by the kernel, and then (ii) relabel the nodes via $\phi$. 

\[ \begin{array}{c}
G \\
(\text{Ker } \phi \triangleleft G) \\
\downarrow q \\
G/\text{Ker } \phi \\
\text{group of cosets} \\
\downarrow \\
\text{Im } \phi \\
\text{any homomorphism} \\
\downarrow i \\
\text{remaining isomorphism ("relabeling")} \\
\end{array} \]
Proof of the FHT

Fundamental homomorphism theorem

If \( \phi: G \to H \) is a homomorphism, then \( \text{Im}(\phi) \cong G / \text{Ker}(\phi) \).

Proof

We will construct an explicit map \( i: G / \text{Ker}(\phi) \to \text{Im}(\phi) \) and prove that it is an isomorphism.

Let \( K = \text{Ker}(\phi) \), and recall that \( G / K = \{aK: a \in G\} \). Define

\[
i: G / K \longrightarrow \text{Im}(\phi), \quad i: gK \longmapsto \phi(g).
\]

- Show \( i \) is well-defined: We must show that if \( aK = bK \), then \( i(aK) = i(bK) \).

Suppose \( aK = bK \). We have

\[
aK = bK \implies b^{-1}aK = K \implies b^{-1}a \in K.
\]

By definition of \( b^{-1}a \in \text{Ker}(\phi) \),

\[
1_H = \phi(b^{-1}a) = \phi(b^{-1})\phi(a) = \phi(b)^{-1}\phi(a) \implies \phi(a) = \phi(b).
\]

By definition of \( i \): \( i(aK) = \phi(a) = \phi(b) = i(bK) \).  

\( \checkmark \)
Proof of FHT (cont.) [Recall: $i: G/K \rightarrow \text{Im}(\phi)$, $i: gK \mapsto \phi(g)$]

Proof (cont.)

- **Show $i$ is a homomorphism**: We must show that $i(aK \cdot bK) = i(aK) i(bK)$.

  $$
  \begin{align*}
  i(aK \cdot bK) &= i(abK) & \text{(}aK \cdot bK := abK\text{)} \\
  &= \phi(ab) & \text{(definition of $i$)} \\
  &= \phi(a) \phi(b) & \text{($\phi$ is a homomorphism)} \\
  &= i(aK) i(bK) & \text{(definition of $i$)}
  \end{align*}
  $$

  Thus, $i$ is a homomorphism.  

- **Show $i$ is surjective (onto)**:

  This means showing that for any element in the codomain (here, $\text{Im}(\phi)$), that some element in the domain (here, $G/K$) gets mapped to it by $i$.

  Pick any $\phi(a) \in \text{Im}(\phi)$. By definition, $i(aK) = \phi(a)$, hence $i$ is surjective.
Proof (cont.)

- **Show $i$ is injective (1–1):** We must show that $i(aK) = i(bK)$ implies $aK = bK$.

Suppose that $i(aK) = i(bK)$. Then

\[
i(aK) = i(bK) \implies \phi(a) = \phi(b) \quad \text{(by definition)}
\]
\[
\implies \phi(b)^{-1} \phi(a) = 1_H \quad \text{(by definition)}
\]
\[
\implies \phi(b^{-1}a) = 1_H \quad \text{($\phi$ is a homom.)}
\]
\[
\implies b^{-1}a \in K \quad \text{(definition of $\text{Ker}(\phi)$)}
\]
\[
\implies b^{-1}aK = K \quad \text{($aH = H \iff a \in H$)}
\]
\[
\implies aK = bK
\]

Thus, $i$ is injective. ⊚

In summary, since $i: G/K \rightarrow \text{Im}(\phi)$ is a well-defined homomorphism that is **injective** (1–1) and **surjective** (onto), it is an **isomorphism**.

Therefore, $G/K \cong \text{Im}(\phi)$, and the FHT is proven. □
Consequences of the FHT

**Corollary**

If \( \phi: G \to H \) is a homomorphism, then \( \text{Im} \phi \leq H \).

**A few special cases**

- If \( \phi: G \to H \) is an embedding, then \( \text{Ker}(\phi) = \{1_G\} \). The FHT says that
  \[ \text{Im}(\phi) \cong G / \{1_G\} \cong G. \]

- If \( \phi: G \to H \) is the map \( \phi(g) = 1_H \) for all \( h \in G \), then \( \text{Ker}(\phi) = G \), so the FHT says that
  \[ \{1_H\} = \text{Im}(\phi) \cong G / G. \]

Let’s use the FHT to determine all homomorphisms \( \phi: C_4 \to C_3 \):

- By the FHT, \( G / \text{Ker} \phi \cong \text{Im} \phi \leq C_3 \), and so \( |\text{Im} \phi| = 1 \) or \( 3 \).
- Since \( \text{Ker} \phi \leq C_4 \), Lagrange’s Theorem also tells us that \( |\text{Ker} \phi| \in \{1, 2, 4\} \), and hence \( |\text{Im} \phi| = |G / \text{Ker} \phi| \in \{1, 2, 4\} \).

Thus, \( |\text{Im} \phi| = 1 \), and so the *only* homomorphism \( \phi: C_4 \to C_3 \) is the trivial one.
How to show two groups are isomorphic

The standard way to show $G \cong H$ is to construct an isomorphism $\phi: G \rightarrow H$.

When the domain is a quotient, there is another method, due to the FHT.

Useful technique

Suppose we want to show that $G/N \cong H$. There are two approaches:

(i) Define a map $\phi: G/N \rightarrow H$ and prove that it is well-defined, a homomorphism, and a bijection.

(ii) Define a map $\phi: G \rightarrow H$ and prove that it is a homomorphism, a surjection (onto), and that $\text{Ker} \phi = N$.

Usually, Method (ii) is easier. Showing well-definedness and injectivity can be tricky.

For example, each of the following are results that we will see very soon, for which (ii) works quite well:

- $\mathbb{Z}/\langle n \rangle \cong \mathbb{Z}_n$;
- $\mathbb{Q}^*/\langle -1 \rangle \cong \mathbb{Q}^+$;
- $AB/B \cong A/(A \cap B)$ (assuming $A, B \triangleleft G$);
- $G/(A \cap B) \cong (G/A) \times (G/B)$ (assuming $G = AB$).
Cyclic groups as quotients

Consider the following normal subgroup of $\mathbb{Z}$:

$$12\mathbb{Z} = \langle 12 \rangle = \{ \ldots, -24, -12, 0, 12, 24, \ldots \} \triangleleft \mathbb{Z}.$$ 

The elements of the quotient group $\mathbb{Z}/\langle 12 \rangle$ are the cosets:

$$0 + \langle 12 \rangle, \; 1 + \langle 12 \rangle, \; 2 + \langle 12 \rangle, \; \ldots, \; 10 + \langle 12 \rangle, \; 11 + \langle 12 \rangle.$$ 

Number theorists call these sets congruence classes modulo 12. We say that two numbers are congruent mod 12 if they are in the same coset.

Recall how to add cosets in the quotient group:

$$(a + \langle 12 \rangle) + (b + \langle 12 \rangle) := (a + b) + \langle 12 \rangle.$$ 

“(The coset containing $a$) + (the coset containing $b$) = the coset containing $a + b$.”

It should be clear that $\mathbb{Z}/\langle 12 \rangle$ is isomorphic to $\mathbb{Z}_{12}$. Formally, this is just the FHT applied to the following homomorphism:

$$\phi : \mathbb{Z} \longrightarrow \mathbb{Z}_{12}, \quad \phi : k \longmapsto k \pmod{12},$$

Clearly, $\text{Ker}(\phi) = \{ \ldots, -24, -12, 0, 12, 24, \ldots \} = \langle 12 \rangle$. By the FHT:

$$\mathbb{Z}/\text{Ker}(\phi) = \mathbb{Z}/\langle 12 \rangle \cong \text{Im}(\phi) = \mathbb{Z}_{12}.$$
A picture of the isomorphism $i: \mathbb{Z}_{12} \rightarrow \mathbb{Z}/\langle 12 \rangle$ (from the VGT website)