Lecture 5.3: Examples of group actions

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Groups acting on elements, subgroups, and cosets

It is frequently of interest to analyze the action of a group G on its elements, subgroups, or cosets of some fixed $H \leq G$.

Sometimes, the orbits and stabilizers of these actions are actually familiar algebraic objects.

Also, sometimes a deep theorem has a slick proof via a clever group action.

For example, we will see how Cayley's theorem (every group G is isomorphic to a group of permutations) follows immediately once we look at the correct action.

Here are common examples of group actions:

- G acts on itself by right-multiplication (or left-multiplication).
- G acts on itself by conjugation.
- G acts on its subgroups by conjugation.
- G acts on the right-cosets of a fixed subgroup $H \leq G$ by right-multiplication.

For each of these, we'll analyze the orbits, stabilizers, and fixed points.

Groups acting on themselves by right-multiplication

We've seen how groups act on themselves by right-multiplication. While this action is boring (any Cayley diagram is an action diagram!), it leads to a slick proof of Cayley's theorem.

Cayley's theorem

If $|G| = n$, then there is an embedding $G \hookrightarrow S_n$.

Proof.

The group G acts on itself (that is, $S = G$) by **right-multiplication**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S) \cong S_n\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto xg.$$

There is only one orbit: $G = S$. The stabilizer of any $x \in G$ is just the identity element:

$$\mathrm{Stab}(x) = \{g \in G \mid xg = x\} = \{e\}\,.$$

Therefore, the kernel of this action is $\mathrm{Ker}\,\phi = \bigcap_{x \in G} \mathrm{Stab}(x) = \{e\}$.

Since $\mathrm{Ker}\,\phi = \{e\}$, the homomorphism ϕ is an embedding. □

Groups acting on themselves by conjugation

Another way a group G can act on itself (that is, $S = G$) is by **conjugation**:

$$\phi\colon G \longrightarrow \text{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

- The orbit of $x \in G$ is its conjugacy class:

$$\text{Orb}(x) = \{x.\phi(g) \mid g \in G\} = \{g^{-1}xg \mid g \in G\} = \text{cl}_G(x)\,.$$

- The stabilizer of x is the set of elements that commute with x; called its centralizer:

$$\text{Stab}(x) = \{g \in G \mid g^{-1}xg = x\} = \{g \in G \mid xg = gx\} := C_G(x)$$

- The fixed points of ϕ are precisely those in the center of G:

$$\text{Fix}(\phi) = \{x \in G \mid g^{-1}xg = x \text{ for all } g \in G\} = Z(G)\,.$$

By the Orbit-Stabilizer theorem, $|G| = |\,\text{Orb}(x)| \cdot |\,\text{Stab}(x)| = |\,\text{cl}_G(x)| \cdot |C_G(x)|$. Thus, we immediately get the following new result about conjugacy classes:

Theorem

For any $x \in G$, the size of the conjugacy class $\text{cl}_G(x)$ divides the size of G.

Groups acting on themselves by conjugation

As an example, consider the action of $G = D_6$ on itself by **conjugation**.

The orbits of the action are the conjugacy classes:

e	r	r^2	f	$r^2 f$	$r^4 f$
r^3	r^5	r^4	rf	$r^3 f$	$r^5 f$

The fixed points of ϕ are the size-1 conjugacy classes. These are the elements in the center: $Z(D_6) = \{e\} \cup \{r^3\} = \langle r^3 \rangle$.

By the Orbit-Stabilizer theorem:

$$|\operatorname{Stab}(x)| = \frac{|D_6|}{|\operatorname{Orb}(x)|} = \frac{12}{|\operatorname{cl}_G(x)|}.$$

The stabilizer subgroups are as follows:

- $\operatorname{Stab}(e) = \operatorname{Stab}(r^3) = D_6$,
- $\operatorname{Stab}(r) = \operatorname{Stab}(r^2) = \operatorname{Stab}(r^4) = \operatorname{Stab}(r^5) = \langle r \rangle = C_6$,
- $\operatorname{Stab}(f) = \{e, r^3, f, r^3 f\} = \langle r^3, f \rangle$,
- $\operatorname{Stab}(rf) = \{e, r^3, rf, r^4 f\} = \langle r^3, rf \rangle$,
- $\operatorname{Stab}(r^i f) = \{e, r^3, r^i f, r^i f\} = \langle r^3, r^i f \rangle$.

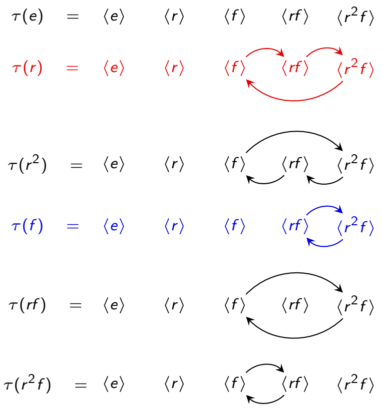
Groups acting on subgroups by conjugation

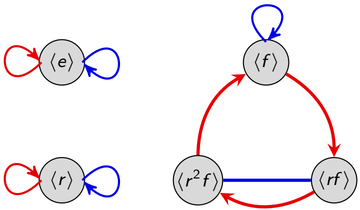
Let $G = D_3$, and let S be the set of proper <u>subgroups</u> of G:

$$S = \Big\{ \langle e \rangle,\ \langle r \rangle,\ \langle f \rangle,\ \langle rf \rangle,\ \langle r^2 f \rangle \Big\}.$$

There is a right group action of $D_3 = \langle r, f \rangle$ on S by conjugation:

$$\tau \colon D_3 \longrightarrow \text{Perm}(S), \qquad \tau(g) = \text{the permutation that sends each } H \text{ to } g^{-1}Hg.$$





The action diagram.

$$\text{Stab}(\langle e \rangle) = \text{Stab}(\langle r \rangle) = D_3 = N_{D_3}(\langle r \rangle)$$
$$\text{Stab}(\langle f \rangle) = \langle f \rangle = N_{D_3}(\langle f \rangle),$$
$$\text{Stab}(\langle rf \rangle) = \langle rf \rangle = N_{D_3}(\langle rf \rangle),$$
$$\text{Stab}(\langle r^2 f \rangle) = \langle r^2 f \rangle = N_{D_3}(\langle r^2 f \rangle).$$

Groups acting on subgroups by conjugation

More generally, any group G acts on its set S of subgroups by **conjugation**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } H \text{ to } g^{-1}Hg.$$

This is a right action, but there is an associated left action: $H \mapsto gHg^{-1}$.

Let $H \leq G$ be an element of S.

- The orbit of H consists of all conjugate subgroups:

$$\mathrm{Orb}(H) = \{g^{-1}Hg \mid g \in G\}\,.$$

- The stabilizer of H is the normalizer of H in G:

$$\mathrm{Stab}(H) = \{g \in G \mid g^{-1}Hg = H\} = N_G(H)\,.$$

- The fixed points of ϕ are precisely the normal subgroups of G:

$$\mathrm{Fix}(\phi) = \{H \leq G \mid g^{-1}Hg = H \text{ for all } g \in G\}\,.$$

- The kernel of this action is G iff every subgroup of G is normal. In this case, ϕ is the trivial homomorphism: pressing the g-button fixes (i.e., normalizes) every subgroup.

Groups acting on cosets of H by right-multiplication

Fix a subgroup $H \leq G$. Then G acts on its **right cosets** by **right-multiplication**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \text{ to } Hxg.$$

Let Hx be an element of $S = G/H$ (the right cosets of H).

- There is only one orbit. For example, given two cosets Hx and Hy,

$$\phi(x^{-1}y) \text{ sends } Hx \longmapsto Hx(x^{-1}y) = Hy.$$

- The stabilizer of Hx is the conjugate subgroup $x^{-1}Hx$:

$$\mathrm{Stab}(Hx) = \{g \in G \mid Hxg = Hx\} = \{g \in G \mid Hxgx^{-1} = H\} = x^{-1}Hx\,.$$

- Assuming $H \neq G$, there are no fixed points of ϕ. The only orbit has size $[G : H] > 1$.
- The kernel of this action is the intersection of all conjugate subgroups of H:

$$\mathrm{Ker}\,\phi = \bigcap_{x \in G} x^{-1}Hx$$

Notice that $\langle e \rangle \leq \mathrm{Ker}\,\phi \leq H$, and $\mathrm{Ker}\,\phi = H$ iff $H \lhd G$.