Lecture 5.7: Finite simple groups

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4120, Modern Algebra
Overview

Definition

A group $G$ is simple if its only normal subgroups are $G$ and $\langle e \rangle$.

Since all Sylow $p$-subgroups are conjugate, the following result is straightforward:

Proposition (HW)

A Sylow $p$-subgroup is normal in $G$ if and only if it is the unique Sylow $p$-subgroup (that is, if $n_p = 1$).

The Sylow theorems are very useful for establishing statements like:

\textit{There are no simple groups of order $k$ (for some $k$).}

To do this, we usually just need to show that $n_p = 1$ for some $p$ dividing $|G|$.

Since we established $n_5 = 1$ for our running example of a group of size $|M| = 200 = 2^3 \cdot 5^2$, there are no simple groups of order 200.
An easy example

Tip
When trying to show that $n_p = 1$, it’s usually more helpful to analyze the largest primes first.

Proposition
There are no simple groups of order 84.

Proof
Since $|G| = 84 = 2^2 \cdot 3 \cdot 7$, the Third Sylow Theorem tells us:

- $n_7$ divides $2^2 \cdot 3 = 12$ (so $n_7 \in \{1, 2, 3, 4, 6, 12\}$)
- $n_7 \equiv_7 1$.

The only possibility is that $n_7 = 1$, so the Sylow 7-subgroup must be normal. □

Observe why it is beneficial to use the largest prime first:

- $n_3$ divides $2^2 \cdot 7 = 28$ and $n_3 \equiv_3 1$. Thus $n_3 \in \{1, 2, 4, 7, 14, 28\}$.
- $n_2$ divides $3 \cdot 7 = 21$ and $n_2 \equiv_2 1$. Thus $n_2 \in \{1, 3, 7, 21\}.$
A harder example

**Proposition**

There are no simple groups of order 351.

**Proof**

Since \(|G| = 351 = 3^3 \cdot 13\), the Third Sylow Theorem tells us:

- \(n_{13}\) divides \(3^3 = 27\) (so \(n_{13} \in \{1, 3, 9, 27\}\))
- \(n_{13} \equiv_{13} 1\).

The only possibilities are \(n_{13} = 1\) or \(27\).

A Sylow 13-subgroup \(P\) has order 13, and a Sylow 3-subgroup \(Q\) has order \(3^3 = 27\). Therefore, \(P \cap Q = \{e\}\).

Suppose \(n_{13} = 27\). Every Sylow 13-subgroup contains 12 non-identity elements, and so \(G\) must contain \(27 \cdot 12 = 324\) elements of order 13.

This leaves \(351 - 324 = 27\) elements in \(G\) not of order 13. Thus, \(G\) contains only one Sylow 3-subgroup (i.e., \(n_3 = 1\)) and so \(G\) cannot be simple. □
The hardest example

Proposition

If \( H \trianglelefteq G \) and \( |G| \) does not divide \([G : H]!\), then \( G \) cannot be simple.

Proof

Let \( G \) act on the right cosets of \( H \) (i.e., \( S = G/H \)) by right-multiplication:

\[
\phi: G \rightarrow \text{Perm}(S) \cong S_n, \quad \phi(g) = \text{the permutation that sends each } Hx \text{ to } Hxg.
\]

Recall that the kernel of \( \phi \) is the intersection of all conjugate subgroups of \( H \):

\[
\text{Ker} \phi = \bigcap_{x \in G} x^{-1}Hx.
\]

Notice that \( \langle e \rangle \leq \text{Ker} \phi \leq H \trianglelefteq G \), and \( \text{Ker} \phi \triangleleft G \).

If \( \text{Ker} \phi = \langle e \rangle \) then \( \phi: G \hookrightarrow S_n \) is an embedding. But this is impossible because \( |G| \) does not divide \( |S_n| = [G : H]! \).

\( \square \)

Corollary

There are no simple groups of order 24.
Theorem (classification of finite simple groups)

Every finite simple group is isomorphic to one of the following groups:

- A cyclic group $\mathbb{Z}_p$, with $p$ prime;
- An alternating group $A_n$, with $n \geq 5$;
- A Lie-type Chevalley group: $\text{PSL}(n, q)$, $\text{PSU}(n, q)$, $\text{PsP}(2n, p)$, and $P\Omega^e(n, q)$;
- A Lie-type group (twisted Chevalley group or the Tits group): $D_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, $^2F_4(2^n)'$, $G_2(q)$, $^2G_2(3^n)$, $^2B(2^n)$;
- One of 26 exceptional “sporadic groups.”

The two largest sporadic groups are the:

- “baby monster group” $B$, which has order
  $$|B| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47 \approx 4.15 \times 10^{33};$$
- “monster group” $M$, which has order
  $$|M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8.08 \times 10^{53}.$$

The proof of this classification theorem is spread across $\approx 15,000$ pages in $\approx 500$ journal articles by over 100 authors, published between 1955 and 2004.
The Periodic Table Of Finite Simple Groups

Dykin Diagrams of Simple Lie Algebras

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Order</th>
<th>M11</th>
<th>M12</th>
<th>M22</th>
<th>M23</th>
<th>M24</th>
<th>(I1), (I11)</th>
<th>H1</th>
<th>H2</th>
<th>H3/M</th>
<th>I4</th>
<th>HS</th>
<th>McL</th>
<th>He</th>
<th>Ru</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>7520</td>
<td>95040</td>
<td>443520</td>
<td>1020960</td>
<td>244823040</td>
<td>1757560</td>
<td>604800</td>
<td>50252960</td>
<td>44352000</td>
<td>899128000</td>
<td>4056387200</td>
<td>14192414400</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Group Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alternating Groups</td>
</tr>
<tr>
<td>Classical Chevalley Groups</td>
</tr>
<tr>
<td>Chevalley Groups</td>
</tr>
<tr>
<td>Classical Steinberg Groups</td>
</tr>
<tr>
<td>Steinberg Groups</td>
</tr>
<tr>
<td>Suzuki Groups</td>
</tr>
<tr>
<td>Ree Groups and Tits Group*</td>
</tr>
<tr>
<td>Sporadic Groups</td>
</tr>
<tr>
<td>Cyclic Groups</td>
</tr>
</tbody>
</table>

*For the simple groups of Lie type, symbolized by the letter G, the following notation is used: G(2q) for G2, G(3q) for G3, etc. A is the Tits group 2A1. B is the Suzuki group 2B2, C is the Ree group 2F4(2). A1 and A2 are the special linear and the special orthogonal groups, respectively.
Musical Fruitcake

Klein Four

Open iTunes to preview, buy, and download music.

<table>
<thead>
<tr>
<th>Name</th>
<th>Artist</th>
<th>Time</th>
<th>Price</th>
<th>View In iTunes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Power of One</td>
<td>Klein Four</td>
<td>5:16</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>2 Finite Simple Group (of Order Two)</td>
<td>Klein Four</td>
<td>3:00</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>3 Three-Body Problem</td>
<td>Klein Four</td>
<td>3:17</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>4 Just the Four of Us</td>
<td>Klein Four</td>
<td>4:19</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>5 Lemma</td>
<td>Klein Four</td>
<td>3:43</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>6 Calculating</td>
<td>Klein Four</td>
<td>4:09</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>7 XX Potential</td>
<td>Klein Four</td>
<td>3:42</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>8 Confuse Me</td>
<td>Klein Four</td>
<td>3:41</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>9 Universal</td>
<td>Klein Four</td>
<td>4:13</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>10 Contradiction</td>
<td>Klein Four</td>
<td>3:48</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>11 Mathematics Paradise</td>
<td>Klein Four</td>
<td>3:51</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>12 Stefanie (The Ballad of Galois)</td>
<td>Klein Four</td>
<td>4:51</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>13 Musical Fruitcake (Pass it Around)</td>
<td>Klein Four</td>
<td>2:50</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
<tr>
<td>14 Abandon Soap</td>
<td>Klein Four</td>
<td>2:17</td>
<td>$0.99</td>
<td>View In iTunes</td>
</tr>
</tbody>
</table>

14 Songs

$9.99
Genres: Pop, Music
Released: Dec 05, 2005
© 2005 Klein Four

Customer Ratings
★★★★★ 13 Ratings