Lecture 7.6: Rings of fractions

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Motivation

Rings allow us to add, subtract, and multiply, but not necessarily divide.

In any ring: if $a \in R$ is not a zero divisor, then $ax = ay$ implies $x = y$. *This holds even if a^{-1} doesn't exist.*

In other words, by allowing "divison" by non zero-divisors, we can think of R as a subring of a bigger ring that contains a^{-1}.

If $R = \mathbb{Z}$, then this construction yields the rational numbers, $\mathbb{Q}$.

If R is an integral domain, then this construction yields the field of fractions of R.

Goal

Given a commutative ring R, construct a larger ring in which $a \in R$ (that's not a zero divisor) has a multiplicative inverse.

Elements of this larger ring can be thought of as fractions. It will naturally contain an isomorphic copy of R as a subring:

$$R \hookrightarrow \left\{ \frac{r}{1} : r \in R \right\}.$$

From $\mathbb{Z}$ to $\mathbb{Q}$

Let's examine how one can construct the rationals from the integers.

There are many ways to write the same rational number, e.g., $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$

Equivalence of fractions

Given $a, b, c, d \in \mathbb{Z}$, with $b, d \neq 0$,

$$\frac{a}{b} = \frac{c}{d} \qquad \text{if and only if} \qquad ad = bc.$$

Addition and multiplication is defined as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

It is not hard to show that these operations are well-defined.

The integers $\mathbb{Z}$ can be identified with the subring $\{\frac{a}{1} : a \in \mathbb{Z}\}$ of $\mathbb{Q}$, and every $a \neq 0$ has a multiplicative inverse in $\mathbb{Q}$.

We can do a similar construction in any commutative ring!

Rings of fractions

Blanket assumptions

- R is a commutative ring.
- $D \subseteq R$ is nonempty, multiplicatively closed $[d_1, d_2 \in D \Rightarrow d_1 d_2 \in D]$, and contains no zero divisors.
- Consider the following set of ordered pairs:

$$\mathcal{F} = \{(r, d) \mid r \in R,\ d \in D\},$$

Define an equivalence relation: $(r_1, d_1) \sim (r_2, d_2)$ iff $r_1 d_2 = r_2 d_1$. Denote this equvalence class containing (r_1, d_1) by $\frac{r_1}{d_1}$, or r_1/d_1.

Definition

The ring of fractions of D with respect to R is the set of equivalence classes, $R_D := \mathcal{F}/\sim$, where

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} := \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \qquad \text{and} \qquad \frac{r_1}{d_1} \times \frac{r_2}{d_2} := \frac{r_1 r_2}{d_1 d_2}.$$

Rings of fractions

Basic properties (HW)

1. These operations on $R_D = \mathcal{F}/\sim$ are well-defined.
2. $(R_D, +)$ is an abelian group with identity $\frac{0}{d}$, for any $d \in D$. The additive inverse of $\frac{a}{d}$ is $\frac{-a}{d}$.
3. Multiplication is associative, distributive, and commutative.
4. R_D has multiplicative identity $\frac{d}{d}$, for any $d \in D$.

Examples

1. Let $R = \mathbb{Z}$ (or $R = 2\mathbb{Z}$) and $D = R - \{0\}$. Then the ring of fractions is $R_D = \mathbb{Q}$.
2. If R is an integral domain and $D = R - \{0\}$, then R_D is a field, called the field of fractions.
3. If $R = F[x]$ and $D = \{x^n \mid n \in \mathbb{Z}\}$, then $R_D = F[x, x^{-1}]$, the Laurent polynomials over F.
4. If $R = \mathbb{Z}$ and $D = 5\mathbb{Z}$, then $R_D = \mathbb{Z}[\frac{1}{5}]$, which are "polynomials in $\frac{1}{5}$" over $\mathbb{Z}$.
5. If R is an integral domain and $D = \{d\}$, then $R_D = R[\frac{1}{d}]$, the set of all "polynomials in $\frac{1}{d}$" over R.

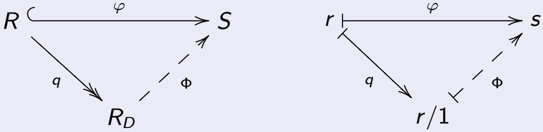
Universal property of the ring of fractions

This says R_D is the "smallest" ring contaning R and all fractions of elements in D:

Theorem

Let S be any commutative ring with 1 and let $\varphi\colon R \hookrightarrow S$ be any ring embedding such that $\phi(d)$ is a unit in S for every $d \in D$.

Then there is a unique ring embedding $\Phi\colon R_D \to S$ such that $\Phi \circ q = \varphi$.



Proof

Define $\Phi\colon R_D \to S$ by $\Phi(r/d) = \varphi(r)\varphi(d)^{-1}$. This is well-defined and 1–1. (HW)

Uniqueness. Suppose $\Psi\colon R_D \to S$ is another embedding with $\Psi \circ q = \varphi$. Then

$$\Psi(r/d) = \Psi((r/1)\cdot(d/1)^{-1}) = \Psi(r/1)\cdot\Psi(d/1)^{-1} = \varphi(r)\varphi(d)^{-1} = \Phi(r/d).$$

Thus, $\Psi = \Phi$. $\square$