Lecture 1.5: Multisets and multichoosing

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Overview

Consider an n-element set S. We can construct:

- lists from S (order matters)
- sets from S (order doesn't matter).

We can count:

- lists of length k:
  - $n(n-1)\cdots(n-k+1) = \dfrac{n!}{(n-k)!}$, if no repetitions allowed
  - n^k, if repetitions are allowed.
- sets of size k:
  - $\dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$, if no repetitions allowed
  - ??? if repetitions are allowed.

In this lecture, we will answer this last part. A set with repetition is called a multiset.

Notation

Definition

Let $\left(\!\binom{n}{k}\!\right)$ be the number of k-element multisets on an n-element set.

We will write multisets as $\langle \dots \rangle$, rather than $\{\dots\}$.

Remark

Unlike for combinations, k could be larger than n.

Exercise

Let $S = \{a, b, c, d\}$.

(i) How many 2-element sets can be formed from S?

(ii) How many 2-element multisets can be formed from S?

Exercise (rephrased)

Let $S = \{a, b, c, d\}$.

(i) How many ways can we choose 2 elements from S?

(ii) How many ways can we multichoose 2 elements from S?

Counting multisets

Proposition

The number of k-element multisets on an n-element set is $\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$.

Proof

We will encode every multiset using "*stars and bars notation*."

Each $*$ represents an element, and the $\big|$ represents a "divider."

Counting multisets

Examples

1. You want to buy 3 hats and there are 5 colors: R, G, B, Y, O. How many possibilities are there?
2. You want to buy 5 hats and there are 3 colors: R, G, B, Y, O. How many possibilities are there?

Counting multisets

Examples

1. How many ways can you buy 6 sodas from a vending maching that has 8 flavors?
2. How many ways can you buy 7 sodas from a vending maching that has 7 flavors?

A multiset identity

Theorem

For any $n, k \geq 1$, we have $\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{k-1}{n-1}\!\!\right)$.

Proof 1 (algebraic)

Write $\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k} = \binom{n+k-1}{n-1} = \left(\!\!\binom{k+1}{n-1}\!\!\right)$. □

Proof 2 (combinatorial)

Switch the roles of bars and stars...

□

Summary

We can count various size-k collections of objects, from a "universe" of n objects.

	repetition allowed	no repetition allowed
Ordered (lists)	n^k	$P(n,k) = \dfrac{n!}{(n-k)!}$
Unordered (sets, multisets)	$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$	$C(n,k) = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Different ways to think about multisets (everyone has their favorite)

The quantity $\left(\!\binom{n}{k}\!\right)$ counts:

- the number of ways to put n identical balls into buckets $B_1, \dots, B_n$.
- the number of ways to distribute k candy bars to n people.
- the number of ways to buy k sodas from a vending machine with n varieties.
- the number of ways to choose k scoops of ice cream from n flavors.
- The number of nonnegative integer solutions to $x_1 + x_2 + \cdots + x_n = k$.
- The number of positive integer sequences $a_1, a_2, \dots, a_k$ where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n$.

Combinatorial proofs: counting things different ways

Sometimes, there are different ways to count the same set of objects.

This can lead to two different formulas that are actually the same; a "*combinatorial identity*."

Verifing an identity by counting a set two different ways is a combinatorial proof, the topic of the next lecture.

But first, we'll see an example of this involving multisets.

Combinatorial proofs: counting things different ways

Example

You have 11 Biographies and 8 Mysteries that you want to arrange on your bookshelf, but no two mysteries can be adjacent to each other. How many different rearrangements are possible?