Lecture 1.6: Combinatorial proofs

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Math 4190, Discrete Mathematical Structures

Overview

Recall that the technique of proving a combinatorial identity by carefully counting a set two distinct ways is called a combinatorial proof.

We have a seen a few of these. In this lecture, we will recall those, and then see some new ones. We'll start with an easy one.

Proposition

For $0 \leq k \leq n$,

$$n! = \binom{n}{k} k!\,(n-k)!.$$

Proof

How many different orderings of the numbers $1, \ldots, n$ are possible?

Examples of combinatorial proofs

Proposition

For $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof

How many ways can we choose k people from a group of n?

Examples of combinatorial proofs

Proposition

For $k, n \geq 0$,

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

Proof

How may ways can we choose k-elements from a set of n, if repetitions are allowed?

Examples of combinatorial proofs

Proposition

For $0 \leq k \leq n$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof

How many subsets of an n-element set are there?

Examples of combinatorial proofs

Theorem

For any $x, y \in \mathbb{N}$ and $n \geq 1$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof

In a class with n students, each student must solve one homework problem. There are x calculus problems and y combinatorics problems to choose from. How many different possible outcomes are there?

Examples of combinatorial proofs

Proposition

For $0 \leq k \leq n$,

$$\sum_{k=0}^{n} \binom{n}{2k} = 2^{n-1}.$$

Proof

How many ways can we select an even number of people from a group of n?

Examples of combinatorial proofs

Proposition

For $0 \leq k \leq n$,

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof

How many ways can we select a committee of k people from a group of n, with one designated chair?

Examples of combinatorial proofs

Proposition

For $0 \leq m \leq k \leq n$,

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

Proof

Given a group of n people, how many ways can we choose a size-k committee and a size-m subcommittee?

Examples of combinatorial proofs

Vandermonde's identity

For $0 \leq m \leq k \leq n$,

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}.$$

Proof

How many ways can we select a size-k committee from a group of m men and n women?