Lecture 2.1: Propositions and logical operators

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Logical propositions

Definition

In logic, a proposition is a sentence to which one and only one of the terms *true* or *false* can be applied.

Examples.

1. "*4 is even.*"
2. "$4 \in \{1, 3, 5\}$."
3. "$2 + 2 = 5$."
4. "*There is intelligent life on other planets.*"
5. "*The following computer program halts.*"

Non-examples.

1. "*Do you understand this concept?*"
2. "*Watch the rest of this video.*"
3. "$x^2 = 2x - 3$."
4. "*This statement is false.*"
5. "*Schrödinger's cat is dead.*"

Logical operations

In algebra, variables are placeholders for numbers, often denoted with x, y, and z.

The most common symbols for logical variables are p, q, and r, and these are placeholders for propositions.

Logical variables can take the values of 0 or 1, which denote *false* and *true*, respectively.

We can combine simple statements into compound ones using words and phrases such as: *and*, *or*, *not*, *if . . . then . . .* , *if and only if*, etc.

Except for *not*, all of these operations act on pairs of propositions.

We will precisely define each of these and introduce standard notation.

We will use the concept of a truth table for each one. This describes the effect of the logical operation on all possible inputs.

Conjunction and disjunction

Definition ("and")

If p and q are propositions, their conjunction, p and q, denoted $p \wedge q$, is defined by the following truth table:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Definition ("or")

If p and q are propositions, their disjunction, p or q, denoted $p \vee q$, is defined by the following truth table:

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Negation

Definition ("not")

If p is a proposition, its negation, not p, denoted $\neg p$, is defined by the following truth table:

p	$\neg p$
0	1
1	0

Sometimes the negation of p is denoted $\overline{p}$, or as $\sim p$.

Conditional statement

Consider the following statements:

(a) I am going to wear my raincoat if it rains.

(b) If I do not pass this class, I will not graduate.

(c) I will be on time for this class provided my car starts.

All of these can be written in the form:

If *Condition*, then *Conclusion*.

Definition

The conditional statement, "if p then q", denoted $p \to q$, is defined by the following truth table:

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Think of a conditional statement as a *guarantee*, i.e., $p \to q$ asks whether that guarantee was kept.

Conditional statement

Example

Suppose I make the guarantee that if you get ≥ 90 on the final exam, then you will get an A in the class.

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

p: "*you get* ≥ 90 *on the final exam*"

q: "*you get an* A *in the class*"

Let's check all four possible pairs of p and q to and verify that $p \to q$ makes sense...

Converse and contrapositive

The order of the condition and conclusion in a conditional proposition matters.

Definition

Given a proposition $p \to q$, the:

- converse of $p \to q$ is the proposition $q \to p$.
- contrapositive of $p \to q$ is the proposition $\neg q \to \neg p$.
- inverse of $p \to q$ is the proposition $\neg p \to \neg q$.
- negation of $p \to q$ is the proposition $\neg(p \to q)$.

Remark

A conditional proposition is:

- not equivalent to its converse, inverse, or negation,
- equivalent to its contrapositive.

p	q	$p \to q$	$q \to p$	$\neg q \to \neg p$	$\neg p \to \neg q$	$\neg(p \to q)$
0	0	1	1	1	1	0
0	1	1	0	1	0	0
1	0	0	1	0	1	1
1	1	1	1	1	1	0

Converse vs. contrapositive

Example

Suppose I make the guarantee that if you get ≥ 90 on the final exam, then you will get an A in the class.

p: "*you get* ≥ 90 *on the final exam*"

q: "*you get an* A *in the class*"

p	q	$p \to q$	$q \to p$	$\neg q \to \neg p$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	0
1	1	1	1	1

Let's consider the converse and the contrapositive of $p \to q$...

Biconditional statement

Definition

The biconditional statement, "p if only if q, denoted $p \leftrightarrow q$, is true precisely when p and q have the same truth values.

p	q	$p \to q$	$q \to p$	$\neg q \to \neg p$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	1	0
1	0	0	1	0	0
1	1	1	1	1	1

It is common to abbreviate "*if and only if*" to "*iff*."

"If" vs. "only if" vs. "iff"

Example

Consider the following three statements:

1. I wear Orange *if* it is Friday.
2. I wear Orange *only if* it is Friday.
3. I wear Orange *if and only if* it is Friday.

1. IF it is Friday, THEN I wear Orange.

   $$\text{Friday} \to \text{Orange}$$

   Being Friday is sufficient for me to wear Orange.

   If I am not wearing Orange, then it is not Friday.

   $$\text{Not Orange} \to \text{Not Friday}$$

2. IF I am wearing Orange, THEN it is Friday.

   $$\text{Orange} \to \text{Friday}$$

   Being Friday is necessary for me to wear Orange.

   If it is not Friday, then I am not wearing Orange.

   $$\text{Not Friday} \to \text{Not Orange}$$

Conditional vs. biconditional

Equivalent to: if p then q

- p implies q
- q follows from p
- p, only if q
- q, if p
- p is sufficient for q
- q is necessary for p

Equivalent to: if q then p

- p follows from q
- q implies p
- p, if q
- q, only if p
- p is necessary for q
- q is sufficient for p

Equivalent to: p if and only if q

- p is necessary and sufficient for q
- p is equivalent to q
- If p, then q, and if q, then p
- If p, then q, and conversely