Lecture 2.6: Propositions over a universe

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Propositions over a universe

Definition

Let U be a nonempty set. A proposition over U is a sentence that contains a variable that can take on any value in U and that has a definite truth value as a result of any such substitution. We may write $p(u)$ to denote "*the truth value of p when we substitute in u*."

Examples

Over the integers:

- $x^2 \geq 0$ (always true; a "tautology")
- $x \geq 0$ (sometimes true)
- $x^2 < 0$ (never true; a "contradiction")

Over the rational numbers:

- $(s-1)(s+1) = s^2 - 1$ (tautology)
- $4x^2 - 3x = 0$
- $y^2 = 2$ (contradiction)

Over the power set 2^S for a fixed set S:

- $(A \neq \emptyset) \vee (A = S)$
- $3 \in A$
- $A \cap \{1, 2, 3\} \neq \emptyset$.

Propositions over a universe

All of the laws of logic that we've seen are valid for propositions over a universe.

For example, if p and q are propositions over $\mathbb{Z}$, then $p \wedge q \Rightarrow q$ because $(p \wedge q) \rightarrow q$ is a tautology, no matter what values we substitute for p and q.

Over $\mathbb{N}$, let $p(n)$ be true if $n < 44$, and $q(n)$ be true if $n < 16$, i.e.,

$$p(n) : n < 44 \qquad \text{and} \qquad q(n) : n < 16.$$

Note that in this case, $q \Rightarrow p \wedge q$.

Definition

If p is a proposition over U, then truth set of p is

$$T_p = \{a \in U \mid p(a) \text{ is true}\}.$$

When p is an equation, we often use the term solution set.

Examples

- Let $S = \{1, 2, 3, 4\}$ and $U = 2^S$. The truth set of the proposition $\{1, 2\} \cap A = \emptyset$ over U is $\{\emptyset, \{3\}, \{4\}, \{3, 4\}\}$.
- Over $\mathbb{Z}$, the truth (solution) set of $4x^2 - 3x = 0$ is $\{0\}$.
- Over $\mathbb{Q}$, the solution set of $4x^2 - 3x = 0$ is $\{0, 3/4\}$.

Compound statements

The truth sets of compound propositions can be expressed in terms of the truthsets of simple propositions.

For example:

$$\begin{aligned} a \in T_{p\wedge q} \quad & \text{iff} \quad a \text{ makes } p \wedge q \text{ true} \\ & \text{iff} \quad a \text{ makes both } p \text{ and } q \text{ true} \\ & \text{iff} \quad a \in T_p \cap T_q. \end{aligned}$$

Truth sets of compound statements

$$\begin{aligned} T_{p\wedge q} &= T_p \cap T_q \\ T_{p\vee q} &= T_p \cup T_q \\ T_{\neg p} &= T_p^c \\ T_{p\leftrightarrow q} &= (T_p \cap T_q) \cup (T_p^c \cap T_q^c) \\ T_{p\rightarrow q} &= T_p^c \cup T_q \end{aligned}$$

Equivalence over U

Definition

Two propositions p and q are equivalent over U if $p \leftrightarrow q$ is a tautology. Equivalently, this means that $T_p = T_q$.

Examples

- $x^2 = 4$ and $x = 2$ are equivalent over $\mathbb{N}$, but non-equivalent over $\mathbb{Z}$.
- $A \cap \{4\} \neq \emptyset$ and $4 \in A$ are equivalent propositions over the power set $2^{\mathbb{N}}$.

We can even relax the condition that the universe U is a set.

For example, consider the universe U of *all sets*. (Not a set!)

Over U, the propositions

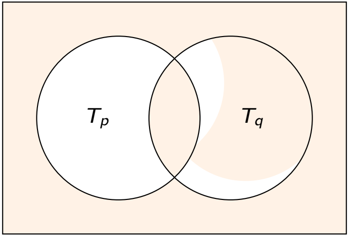
$$p(A, B) : A \subseteq B \qquad \text{and} \qquad q(A, B) : A \cap B = A$$

are equivalent.

Implication over U

Definition

If p and q are propositions over U, then p implies q if $p \to q$ is a tautology.



Examples

- Over the natural numbers: $n \leq 16 \Rightarrow n \leq 44$, because $\{0, 1, \ldots, 16\} \subseteq \{0, 1, \ldots, 44\}$.
- Over the power set $2^{\mathbb{Z}}$: $|A^c| = 1$ implies $A \cap \{0, 1\} \neq \emptyset$.
- Over $2^{\mathbb{Z}}$: $A \subseteq$ even integers $\Rightarrow A \cap$ odd integers $= \emptyset$.