# Lecture 4.3: Partially ordered sets

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Math 4190, Discrete Mathematical Structures

# Recall the basic concepts

## Definition

A partial order on a set $P$ is a relation $\preceq$ that is

(i) reflexive,

(ii) transitive,

(iii) antisymmetric.

We say that $(P, \preceq)$ is a partially ordered set, or a poset.

## Definition (alternate)

A (strict) partial order on a set $P$ is a relation $\prec$ that is

(i) irreflexive,

(ii) transitive,

(iii) antisymmetric

## Definition

If $(P, \preceq)$ is a poset, and for every $a \neq b \in P$, either $a \preceq b$, or $b \preceq a$, then it is a totally ordered set.

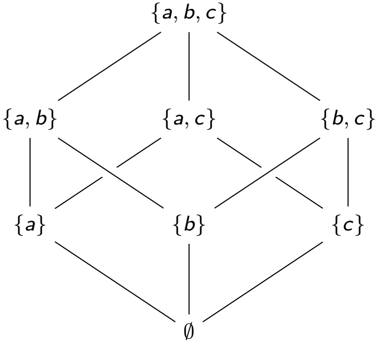
# A bunch of definitions

Let $(P, \preceq)$ be a poset, and $a, x, y, z \in P$.

- If $x \not\preceq y$ and $y \not\preceq x$, then $x$ and $y$ are incomparable. Otherwise, they are comparable.
- If $x \preceq z$, but $\nexists y \in P$ such that $x \prec y \prec z$, then $z$ covers $x$.
- If $a \in P$ but $\nexists x \in P$ such that $x \prec a$, then $a$ is a minimal element.
- If $a \preceq x$ for all $x \in P$, then $a$ is the minimum element.
- If $z \in P$ but $\nexists x \in P$ such that $z \prec x$, then $z$ is a maximal element.
- If $x \preceq z$ for all $x \in P$, then $z$ is the maximum element.
- A chain in a poset is a subset $C \subseteq P$ such that any two elements are comparable.
- An antichain in a poset is a subset $A \subseteq P$ of incomparable elements.
- A poset $(P, \leq_{P'})$ is an extension of $(P, \leq_P)$ if $P = P'$ and $a \leq_P b$ implies $a \leq_{P'} b$.
- A linear extension of a poset is an extension that is a total order.

# Examples of posets

1. Power set: $(2^S, \subseteq)$.
2. Partitions of $[n] = \{1, \ldots, n\}$.
3. Any acyclic directed graph.
4. Divisors of n. Or the integers, by divisibility: $(\mathbb{Z}^+, |)$.
5. Vertices in a rooted tree (e.g., computer directory structure, phylogenetic tree).
6. Strongly connected components in a directed graph.



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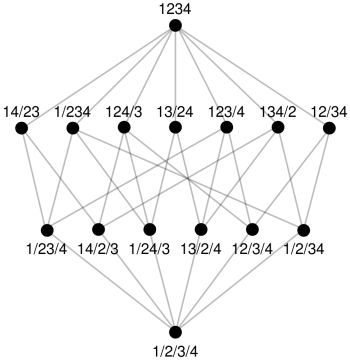


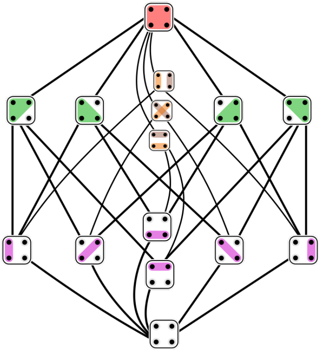

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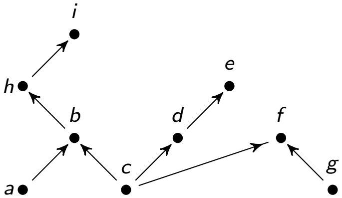


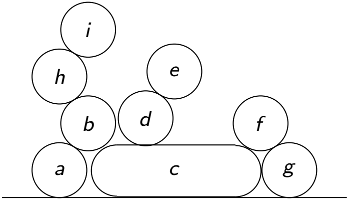

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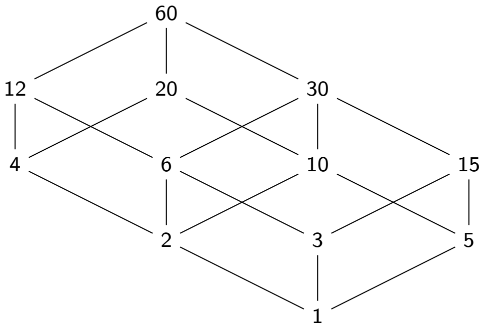

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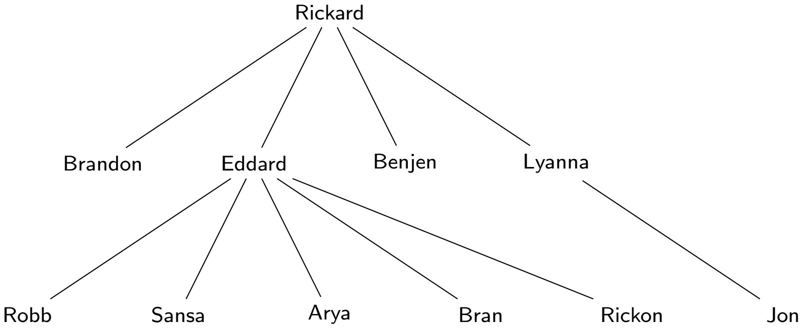

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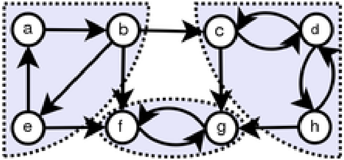
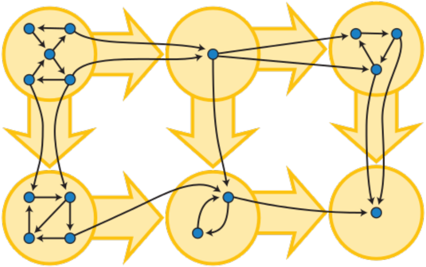


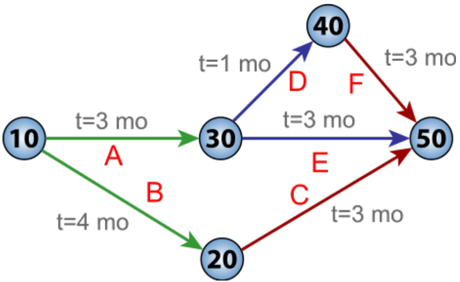



# Applications to scheduling problems

The field of operations research deals with methods (algorithms, optimization, heuristics, etc.) that improve complex decision making.

In the 1950s, the Program Evaluation and Review Technique (PERT) was developed by the U.S. Navy when they were building the Polaris submarine.

Around this time, the Critical Path Method (CPM) was developed by the DuPont chemical company for scheduling maintenance. It was later used during the construction of the World Trade Center.



# Applications to scheduling: PERT and CPM

In both PERT and CPM, the set of scheduled tasks forms a partially ordered set.

The tasks are labeled with the duration that they take to complete.

The shortest possible completion time is given by a maximal chain called a critical path.