Lecture 4.4: Functions

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

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What is a function?

Definition

A function from a set A to a set B is a relation $f \subseteq A \times B$, such that every $a \in A$ is related to exactly one $b \in B$. For notation, we often abbreviate $(a, b) \in f$ as $f(a) = b$.

We call A the domain, B the co-domain, and write $f \colon A \to B$.

The image (or *range*) of f is the set

$$f(A) = \{b \in B \mid b = f(a) \text{ for some } a \in A\} = \{f(a) \mid a \in A\}.$$

The preimage of $b \in B$ is the set

$$f^{-1}(b) := \{a \in A \mid f(a) = b\}.$$

- Sometimes a function is not well-defined, especially if the domain is a set of equivalence classes. For example:

$$f \colon \mathbb{Q} \longrightarrow \mathbb{Z}, \qquad f(\tfrac{m}{n}) = m.$$

- Sometimes functions appear superficially different, but are the same. For example:

$$f, g \colon \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3, \qquad f(x) = x^3, \qquad g(x) = x.$$

- The notation $f^{-1}(b)$ does ***not*** imply that f has an "inverse function".

Ways to describe functions

- Arrow diagrams. (When A and B are finite and small.)
- Formulas (Not always possible.) For example,

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad f(x) = x^2.$$

- Cases. For example, consider

$$f\colon \mathbb{N}^+ \longrightarrow \mathbb{Q}, \qquad f = \{(1,2), (2, \tfrac{1}{2}), (3,9), (4, \tfrac{1}{4}), \dots\},$$

  which can be written as

$$f(x) = \begin{cases} x^2 & x \text{ odd} \\ 1/x & x \text{ even}. \end{cases}$$

- Data (no pattern). A survey of 1000 people asking how many hours of sleep they get in a day is a function

$$f\colon \{0,1,2,\dots,24\} \longrightarrow \{0,1,2,\dots,1000\}.$$

  Or we could "turn it around", as $g\colon \{0,1,2,\dots,1000\} \longrightarrow \{0,1,2,\dots,24\}$.

- Sequences. (If domain is discrete.) For example, $a_n = \frac{1}{n}$.
- Tables. We've seen these for "Boolean" functions, $f\colon \{0,1\}^n \to \{0,1\}$.

Examples of functions

- Let X be any set. The identity function is defined as

$$i\colon X \longrightarrow X, \qquad i(x) = x.$$

- Fix a finite set S. Consider the following "size function" on the power set:

$$f\colon 2^S \longrightarrow \mathbb{N}, \qquad f(A) = |A|.$$

- Let $\mathbb{Z}_2 = \{0,1\}$. The logical OR function, in "polynomial form", is

$$f\colon \mathbb{Z}_2^2 \longrightarrow \mathbb{Z}_2, \qquad f(x,y) = xy + x + y \pmod 2.$$

- Sequences are functions. For example, the sequence $1, 4, 9, 16, \dots$ is

$$f\colon \mathbb{N}^+ \longrightarrow \mathbb{N}^+, \qquad f(n) = n^2.$$

- Let S be a set. Each subset $A \subseteq S$ has a characteristic or indicator function

$$\chi_A\colon S \longrightarrow \{0,1\}, \qquad \chi_A(s) = \begin{cases} 1 & s \in A \\ 0 & s \notin A. \end{cases}$$

- Hash functions from computer science.

Basic properties of functions

Given a function $f\colon X \to Y$ and $A \subseteq X$, we can define the image of A under f:

$$f(A) = \big\{ f(a) \mid a \in A \big\}.$$

Lemma

Let $f\colon X \to Y$. Then for any $A, B \subseteq X$,

(i) $f(A \cup B) \subseteq f(A) \cup f(B)$.

(ii) $f(A \cap B) \subseteq f(A) \cap f(B)$.

Proof

Equality actually holds for one of these... can you figure out which one?

More on sequences

Sequences are just functions from a discrete set, usually $\mathbb{N}$ or $\mathbb{N}^+$.

For example, consider the sequence

$$1, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{4}, \tfrac{1}{5}, \ldots$$

We can express this several ways, depending on whether we start at 0 or 1:

$$f\colon \{0, 1, 2 \ldots\} \to \mathbb{Q}, \quad f(n) = \frac{(-1)^n}{n+1}, \qquad \text{or} \qquad g\colon \{1, 2, \ldots\} \to \mathbb{Q}, \quad g(n) = \frac{(-1)^{n+1}}{n}.$$

For ease of notation, we often define $a_n := f(n)$.

A few more definitions

Definition

Let $f\colon X \to Y$ be a function. Then

- f is injective, or 1–1, if $f(x) = f(y)$ implies $x = y$.
- f is surjective, or onto, if $f(X) = Y$.
- f is bijective if it is both 1–1 and onto.

If $f\colon X \to Y$ is bijective, then we can define its inverse function

$$f^{-1}\colon Y \longrightarrow X, \qquad f^{-1} = \big\{(b,a) \mid (a,b) \in f\big\}.$$

Given $f\colon X \to Y$ and $g\colon Y \to Z$, we can define the composition

$$g \circ f\colon X \longrightarrow Z, \qquad g \circ f = \big\{(x,z) \mid \exists y \in Y \text{ such that } (x,y) \in f,\ (y,z) \in g\big\}.$$

Definition

Two sets X, Y have the same cardinality (size) if there exists a bijection $f\colon X \to Y$.

Injective (1–1) iff left-cancelable

Definition

Suppose $f\colon Y \to Z$, and $g_1, g_2\colon X \to Y$. Then f is left-cancelable if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

Theorem

A function is left-cancelable iff it is injective.

Proof

Surjective (onto) iff right-cancelable

Theorem

Suppose $f\colon X \to Y$, and $h_1, h_2\colon Y \to Z$. Then f is right-cancelable if $h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$

Theorem

A function is right-cancelable iff it is surjective.

Proof