TOPICS: LINEAR MAPS, INNER PRODUCTS, AND ORTHOGONALITY

- 1. Let $T: V \to W$ be a linear map between vector spaces. Prove that $\ker(T) := \{v \in V \mid T(v) = 0\}$ is a subspace of V.
- 2. Let $V = \mathcal{C}^1(\mathbb{R})$, the vector space of differentiable real-valued functions. Consider the linear operator $T = \frac{d}{dt} + 3$.
 - (a) The kernel of T can be characterized precisely by the set of all functions that solve a particular differential equation. Write down this equation.
 - (b) Find the general solution for the differential equation you found in Part (a), and hence an explicit formula for ker(T).
 - (c) Write down an explicit basis for the solution space, ker(T). What is the dimension of this vector space?
- 3. Let $\mathbf{v} = (3, 4) \in \mathbb{R}^2$.
 - (a) Compute $||\mathbf{v}|| := \sqrt{\mathbf{v} \cdot \mathbf{v}}$.
 - (b) Recall that $\{\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)\}$ is an *orthonormal basis* for \mathbb{R}^2 . Decompose \mathbf{v} into this basis, i.e., write $\mathbf{v} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ for some $a_1, a_2 \in \mathbb{R}$.
 - (c) Sketch \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{v} in \mathbb{R}^2 . Graphically show what a_1 and a_2 represent in terms of the projection of \mathbf{v} onto the unit vectors \mathbf{e}_1 and \mathbf{e}_2 .
 - (d) The set $\{\mathbf{v}_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), \mathbf{v}_2 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\}$ is also an orthonormal basis. Decompose \mathbf{v} into this basis, i.e., write $\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$ for some $b_1, b_2 \in \mathbb{R}$.
 - (e) On a new set of axes, sketch \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v} in \mathbb{R}^2 . Graphically show what b_1 and b_2 represent in terms of the projection of \mathbf{v} onto the unit vectors \mathbf{v}_1 and \mathbf{v}_2 .
- 4. For this problem, consider the vector space $V = \mathbb{R}^3$ and use the vector dot product as the inner product.
 - (a) Show that the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = (1, 2, -2), \quad \mathbf{v}_2 = (0, 1, 1), \quad \mathbf{v}_3 = (-4, 1, -1).$$

is an orthogonal set, but not orthonormal.

(b) Normalize \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 to get an *orthonormal* basis of \mathbb{R}^3 . That is, compute the following:

$$\mathcal{B} = \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\} \quad \text{where} \quad \mathbf{n}_i = rac{\mathbf{v}_i}{||\mathbf{v}_i||}.$$

(c) Use the dot product to express the vector $\mathbf{w} = (1, 2, 3)$ in terms of \mathbf{n}_1 , \mathbf{n}_2 , and \mathbf{n}_3 . That is, find C_1 , C_2 , and C_3 such that

$$\mathbf{w} = C_1 \mathbf{n}_1 + C_2 \mathbf{n}_2 + C_3 \mathbf{n}_3$$

5. Let $\mathbb{R}_3[x] = \{a_3x^3 + a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R}\}$, the vector space of polynomials of degree at most 3. Define the following *inner product* on $\mathbb{R}_3[x]$:

$$\langle f,g \rangle = \int_{-1}^{1} f(x)g(x) \, dx$$

- (a) Verify that this is indeed an inner product on $\mathbb{R}_3[x]$.
- (b) Consider the two sets

$$\mathcal{B}_1 = \{1, x, 3x^2 - 1, 5x^3 - 3x\}, \qquad \mathcal{B}_2 = \{1, x, x^2, x^3\}$$

that are both bases for $\mathbb{R}_3[x]$. Show that \mathcal{B}_1 is an orthogonal set, but \mathcal{B}_2 is not. (The set \mathcal{B}_1 are the first four *Legendre polynomials*, $P_n(x)$ for $n = 0, \ldots, 3$. When we study Sturm-Liouville theory, we will see why the Legendre polynomials are always orthogonal!)

- (c) For each $f \in \mathcal{B}_1$, compute the norm of f, which is defined as $||f|| = \langle f, f \rangle^{1/2}$. Find an orthonormal basis for $\mathbb{R}_3[x]$ by normalizing the elements in \mathcal{B}_1 .
- (d) Consider the polynomial $f(x) = 3x^3 2x^2 + 4$. Use orthogonality to write f(x) using the elements in \mathcal{B}_1 . That is, find C_0 , C_1 , C_2 , and C_3 such that

$$3x^3 - 2x^2 + 4 = C_0 + C_1x + C_2(3x^2 - 1) + C_3(5x^3 - 3x).$$