Topics: Real Fourier series, and Fourier sine & cosine series

- 1. Find the Fourier series of the following functions without computing any integrals.
  - (a)  $f(x) = 2 3\sin 4x + 5\cos 6x$ ,
  - (b)  $f(x) = \sin^2 x$ . [Hint: Use a standard trig identity.]
- 2. Consider the sawtooth wave defined on [-1,1] by the function f(t)=t, and extended to be periodic of period T=2.
  - (a) Sketch the graph of f(t) on [-7, 7].
  - (b) Compute the Fourier series of f(t).
  - (c) The differential equation

$$x''(t) + \omega^2 x(t) = f(t)$$

describes the motion of a simple harmonic oscillator, subject to a driving force given by the sawtooth wave f(t). Find the general solution by first solving the homogeneous equation, and then looking for a particular solution of the form

$$x_p(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t).$$

3. Consider the  $2\pi$ -periodic function defined on  $[-\pi, \pi]$  by

$$f(t) = \begin{cases} 0 & -\pi \le t < 0, \\ t & 0 \le t \le \pi, \end{cases}$$

- (a) Sketch the graph of f(t) on  $[-7\pi, 7\pi]$ .
- (b) Compute the Fourier series of f(t).
- (c) Sketch the graph of the resulting Fourier series. [It will be the same as the answer to Part (a) except at the points of discontinuity.]
- (d) Solve the differential equation  $x''(t) + \omega^2 x(t) = f(t)$ . Look for a particular solution of the form

$$x_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt.$$

4. Determine which of the following functions are even, which are odd, and which are neither.

(a)  $f(x) = x^3 + 3x$ 

(e)  $f(x) = \frac{1}{x}$ 

(b)  $f(x) = 4\sin 2x$ 

(f)  $f(x) = \frac{1}{2}(e^x + e^{-x})$ 

(c)  $f(x) = x^2 + |x|$ 

(g)  $f(x) = x \cos x$ 

(d)  $f(x) = e^x$ 

(h)  $f(x) = \frac{1}{2}(e^x - e^{-x}).$ 

- 5. In this problem, we will investigate why in many Fourier series, every other coefficient is zero. This has to do with certain symmetries in the graph.
  - (a) The Fourier series of an odd function consists only of sine-terms. What additional symmetry conditions on f will imply that the sine coefficients with even indices will be zero (i.e., each  $b_{2n} = 0$ )? Give an example of a non-zero function satisfying this additional condition.
  - (b) What symmetry conditions on f will imply that the sine coefficients with odd indices will be zero (i.e., each  $b_{2n+1} = 0$ )? Give an example of a non-zero function satisfying this additional condition.
  - (c) Sketch the graph of a non-zero even function, such that  $a_{2n} = 0$  for all n.
  - (d) Sketch the graph of a non-zero even function, such that  $a_{2n+1} = 0$  for all n.
- 6. Consider the function  $f(x) = x^2$  defined on the interval [0, L]. For this problem, you will determine the Fourier series, Fourier cosine series, and Fourier sine series of f(x). Feel free to use a computer to find any indefinite integrals that you need.
  - (a) Sketch the even extension of f and compute its Fourier cosine series.
  - (b) Sketch the odd extension of f and compute its Fourier sine series.
  - (c) Sketch the periodic extension of f and compute its Fourier series.