

TOPICS: PDES ON UNBOUNDED DOMAINS. LAPLACE & FOURIER TRANSFORMS.

1. Recall that the solution to the initial value problem for the heat equation, where  $x \in \mathbb{R}$  and  $t > 0$ :

$$u_t = c^2 u_{xx}, \quad u(x, 0) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \quad \text{is} \quad u(x, t) = \int_{-\infty}^{\infty} h(y) \frac{1}{\sqrt{4\pi c^2 t}} e^{-(x-y)^2/4c^2 t} dy,$$

where  $h(x) = u(x, 0)$ . Sketch the heat distribution at  $t = 0$  and several larger values of  $t$  on the same set of axes. Then change variables by setting  $z = (y - x)/\sqrt{4c^2 t}$  and write the above solution in terms of the Gauss error function,  $\text{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-r^2} dr$ .

2. Starting with *d'Alembert's formula*, the general solution to the wave equation on  $x \in \mathbb{R}$ , solve the following initial value problem:

$$u_{tt} = c^2 u_{xx}, \quad u(x, 0) = \frac{1}{1 + x^2/4}, \quad u_t(x, 0) = 0.$$

Then repeat for the following IVP:

$$u_{tt} = c^2 u_{xx}, \quad u(x, 0) = 0, \quad u_t(x, 0) = \frac{1}{1 + x^2/4}.$$

Describe what each IVP models. Compare and contrast the differences in the solutions. For each one, sketch the function  $u(x, t)$  for  $t = 0$ , and several larger values of  $t$  on the same set of axes.

3. Solve the I/BVP for the heat equation on the semi-infinite domain  $x > 0$ ,  $t > 0$ :

$$u_t = c^2 u_{xx}, \quad u(0, t) = 0, \quad u(x, 0) = 1.$$

Write your answer in terms of the erf function. Sketch a graph of  $u(x, t)$  at  $t = 0$  and several larger values of  $t$ . What is the long-term behavior?

4. The following I/BVP models what happens to a falling cable that is lying on a table that is suddenly removed:

$$u_{tt} = c^2 u_{xx} - g, \quad u(0, t) = 0, \quad u(x, 0) = u_t(x, 0) = 0.$$

Assume that the domain of  $u(x, t)$  is  $x > 0$  and  $t > 0$ . Solve this PDE using Laplace transforms and draw several time snapshots of the solution.

5. Use a Fourier transform to solve the following initial value problem for the inhomogeneous heat equation, where  $x \in \mathbb{R}$  and  $t > 0$ :

$$u_t = u_{xx} + h(x, t), \quad u(x, 0) = 0.$$

6. Use a Fourier transform to solve the following initial value problem for the free Schrödinger equation, where  $x \in \mathbb{R}$  and  $t > 0$ :

$$u_t = iu_{xx}, \quad u(x, 0) = e^{-x^2}.$$