TOPICS: PDEs in other coordinate systems

In the process of solving these problems, you will encounter several ODEs, Sturm-Liouville problems, PDEs, and Fourier series, many of which you have encountered before. You do not need to re-derive the solutions of anything you have previously solved.

1. Let $u(r, \theta)$ be a function defined on the disk of radius R. Consider the following boundary value problem for Laplace's equation in polar coordinates:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$
 $u(r, \theta + 2\pi) = u(r, \theta),$ $u(R, \theta) = 2 - 3\cos\theta + 5\sin 2\theta.$

- (a) Assume that a solution has the form $u(r, \theta) = R(r)T(\theta)$. Plug this back in and separate variables to get an equation for R and T, including boundary conditions.
- (b) Solve the ODEs for R(r) and $T(\theta)$, and determine all possible eigenvalues λ_n . Make sure to impose the additional requirement that R(0) exists.
- (c) Find the general solution to Laplace's equation in polar coordinates.
- (d) Plug in r = R to find the particular solution to this boundary value problem.
- 2. Let $u(r, \theta, t)$ be a function defined on the disk of radius R = 1, and for all $t \ge 0$. Consider the following initial/boundary value problem for the *heat equation* in polar coordinates:

$$u_t = c^2 \Delta u,$$
 $u(r, \theta + 2\pi) = u(r, \theta),$ $u(r, \theta, 0) = 1 - r^2 + h(r, \theta)$
 $u(1, \theta, t) = 2 - 3\cos\theta + 5\sin 2\theta.$

- (a) Find $h(r,\theta)$, the steady-state solution.
- (b) Make the change of varibles $v(r, \theta, t) = u(r, \theta, t) h(r, \theta)$, and re-write the PDE above, including the boundary and initial conditions, in terms of v instead of u.
- (c) Find the general solution for this homogeneous PDE using separation of variables. Assume that $v(r, \theta, t) = f(r, \theta)g(t)$.
- (d) Find the particular solution that satisfies the initial condition.
- 3. Let $u(r, \theta, t)$ be a function defined on the disk of radius R = 1, and for all $t \ge 0$. Consider the following initial/boundary value problem for the wave equation in polar coordinates:

$$u_{tt} = c^2 \Delta u,$$
 $u(r, \theta + 2\pi) = u(r, \theta)$ $u(r, \theta, 0) = 1 - r^2$
 $u(1, \theta, t) = 0,$ $u_t(r, \theta, 0) = 0.$

- (a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of the boundary conditions and both initial conditions.
- (b) Assume there is a solution of the form $u(r, \theta, t) = f(r, \theta)g(t)$. Plug this back in and separate variables to get a BVP for f and an IVP for g.
- (c) Find the general solution to this BVP for u.
- (d) Find the particular solution that additionally satisfies the initial conditions.

4. Let $u(r, \theta, \phi, t)$ be the temperature of a sphere of radius $R = \pi$. Assume that the initial temperature is constant, and that temperature does not depend on latitude or longitude. In this case, $u(r, \theta, \phi, t) = u(r, t)$, and the heat equation reduces to

$$u_t = c^2(u_{rr} + \frac{2}{r}u_r), \qquad u(\pi, t) = 0, \qquad u(r, 0) = T_0.$$

There is an implied boundary condition at r = 0, that u(0, t) is finite.

- (a) Assume that there is a solution of the form u(r,t) = f(r)g(t). Separate variables to get two equation, and ODE for g, and a (singular) Sturm-Liouville problem for f.
- (b) Solve the Sturm-Liouville problem for f. [Hint: One could use the power series method, but a much easier way is to define y(r) = rf(r), and re-write the problem in terms of y.]
- (c) Find the general solution to this PDE.
- (d) Find the particular solution that additionally satisfies the initial condition. Leave the formulas for coefficients in terms of integrals; no need to solve them.
- 5. Consider a sphere of radius R = 1, and suppose that $u(r, \phi)$ represents a potential that depends only on the radius $r \in [0, 1]$ and latitude $\phi \in [0, \pi]$. In this problem, we will solve Laplace's equation under these conditions. The boundary value problem becomes

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\sin\phi}(u_\phi\sin\phi)_\phi = 0, \qquad u(1,\phi) = f(\phi).$$

- (a) Assume $u(r, \phi) = R(r)Y(\phi)$. Plug this back in and separate variables by multiplying both sides by r^2/RY .
- (b) Change variables by letting $x = \cos \phi$ for -1 < x < 1 and $y(x) = Y(\arccos(x)) = Y(\phi)$. Then derive the following equations:

$$-((1-x^2)y')' = \lambda y,$$
 $(r^2R')' = \lambda R,$

(c) The equation for y(x) should look familiar – it is Legendre's equation (see HW 2, 4, and 8). Recall that it has bounded, continuous solutions on [-1, 1] when

$$\lambda_n = n(n+1), \quad y_n(x) = P_n(x), \quad n = 0, 1, 2, \dots$$

These are the eigenvalues and eigenfunctions of the related (singular) Sturm-Liouville problem (see HW 8). Carry out the details of deriving the general solution to Laplace's equation, which will be

$$u(r,\phi) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\phi), \quad \text{where} \quad c_n = \frac{1}{||P_n||^2} \int_0^{\pi} f(\phi) P_n(\cos\phi) \sin\phi \, d\phi.$$

(d) If $f(\phi) = \sin \phi$, find an approximate, fourth-order solution. That is, truncate it after the n = 4 term. The Legendre polynomials can be derived from the formula

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} \left[(x^2 - 1)^n \right],$$

but feel free to look them up online for this part.