Lecture 4.4: Sturm-Liouville theory

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Definition

A Sturm-Liouville equation is a 2nd order ODE of the following form:

$$-\frac{d}{dx}\Big(p(x)y'\Big) + q(x)y = \lambda w(x)y, \qquad \text{where } p(x),\ q(x),\ w(x) > 0.$$

We are usually interested in solutions $y(x)$ on a bounded interval $[a,b]$, under some homogeneous BCs:

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) &= 0 \qquad & \alpha_1^2 + \alpha_2^2 > 0 \\ \beta_1 y(b) + \beta_2 y'(b) &= 0 \qquad & \beta_1^2 + \beta_2^2 > 0. \end{aligned}$$

Together, this BVP is called a Sturm-Liouville (SL) problem.

Remark

Consider the linear differential operator $L = \frac{1}{w(x)}\Big(-\frac{d}{dx}\Big[p(x)\frac{d}{dx}\Big] + q(x)\Big)$.

$$\mathbb{C}^\infty[a,b] \xrightarrow{L_1 = p(x)\frac{d}{dx}} \mathbb{C}^\infty[a,b] \xrightarrow{L_2 = -\frac{1}{w(x)}\frac{d}{dx} + \frac{q(x)}{w(x)}} \mathbb{C}^\infty[a,b]$$

$$y \longmapsto p(x)y'(x) \longmapsto \frac{-1}{w(x)}\frac{d}{dx}\big[p(x)y'(x)\big] + \frac{q(x)}{w(x)}y(x)$$

An SL equation is just an eigenvalue equation: $Ly = \lambda y$, and $L = L_2 \circ L_1$ is self-adjoint!.

Self-adjointness of the SL operator

Theorem

The SL operator $L = \frac{1}{w(x)}\Big(-\frac{d}{dx}\Big[p(x)\frac{d}{dx}\Big] + q(x)\Big)$ is self-adjoint on $\mathcal{C}^{\infty}_{\alpha,\beta}[a,b]$ with respect to the inner product

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}w(x)\,dx.$$

Proof

Main theorem

The Sturm-Liouville problem $-(p(x)y')' + q(x)y = \lambda w(x)y$ subject to the homogeneous BCs

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \qquad \alpha_1^2 + \alpha_2^2 > 0$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0 \qquad \beta_1^2 + \beta_2^2 > 0.$$

has:

- infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 \cdots \to \infty$;
- An orthonormal basis of eigenvectors $\{y_n\}$, so that every $f \in \mathcal{C}^\infty_{\alpha,\beta}[a,b]$ can be written uniquely as

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x).$$

Remarks

Every 2nd order linear homogeneous ODE, $y'' + P(x)y' + Q(x)y = 0$ can be written as a Sturm-Liouville equation, called its self-adjoint form.

Goal

Given a Sturm-Liouville problem $Ly = \lambda y$ (with BCs):

- Find its eigenvalues.
- Find its eigenfunctions (which are orthogonal!).

Some familiar examples

Definition (recall)

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$$\beta_1 y(b) + \beta_2 y'(b) = 0 \qquad \beta_1^2 + \beta_2^2 > 0.$$

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Example 1 (Dirichlet BCs)

$-y'' = \lambda y, \quad y(0) = 0, \quad y(L) = 0$ is a SL problem.

Here, $p(x) = 1$, $q(x) = 0$, $w(x) = 1$, $\alpha_1 = \beta_1 = 1$, and $\alpha_2 = \beta_2 = 0$.

- Eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$
- Eigenfunctions: $y_n(x) = \sin(\frac{n\pi x}{L})$.

Some familiar examples

Definition (recall)

A Sturm-Liouville equation is a 2nd order ODE of the following form:

$$-(p(x)y')' + q(x)y = \lambda w(x)y, \qquad \text{where } p(x),\ q(x),\ w(x) > 0.$$

We are usually interested in solutions $y(x)$ on a bounded interval $[a, b]$, under some homogeneous BCs:

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) &= 0 \qquad \alpha_1^2 + \alpha_2^2 > 0 \\ \beta_1 y(b) + \beta_2 y'(b) &= 0 \qquad \beta_1^2 + \beta_2^2 > 0. \end{aligned}$$

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Example 2 (Neumann BCs)

$-y'' = \lambda y, \quad y'(0) = 0, \quad y'(L) = 0$ is a SL problem.

Here, $p(x) = 1$, $q(x) = 0$, $w(x) = 1$, $\alpha_1 = \beta_1 = 0$, and $\alpha_2 = \beta_2 = 1$.

- Eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, 3, \dots$
- Eigenfunctions: $y_n(x) = \cos(\frac{n\pi x}{L})$.

Some familiar examples

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$$\beta_1 y(b) + \beta_2 y'(b) = 0 \qquad \beta_1^2 + \beta_2^2 > 0.$$

Together, this BVP is called a Sturm-Liouville (SL) problem.

Example 3 (Mixed BCs)

$-y'' = \lambda y, \quad y(0) = 0, \quad y'(L) = 0$ is a SL problem.

Here, $p(x) = 1$, $q(x) = 0$, $w(x) = 1$, $\alpha_1 = \beta_2 = 1$, and $\alpha_2 = \beta_1 = 0$.

- Eigenvalues: $\lambda_n = \left(\frac{(n+0.5)\pi}{L}\right)^2, \quad n = 0, 1, 2, 3, \dots$
- Eigenfunctions: $y_n(x) = \sin\left(\frac{(n+0.5)\pi x}{L}\right)$.

Some familiar examples

Definition (recall)

A Sturm-Liouville equation is a 2nd order ODE of the following form:

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Example 4 (Robin BCs)

$-y'' = \lambda y, \quad y(0) = 0, \quad y(L) + y'(L) = 0$ is a SL problem.

Here, $p(x) = 1$, $q(x) = 0$, $w(x) = 1$, $\alpha_1 = \beta_1 = \beta_2 = 1$, and $\alpha_2 = 0$.

- Eigenvalues: $\lambda_n = \omega_n^2, \quad n = 1, 2, \dots$ [ω_n's are the positive roots of $y(x) = x - \tan Lx$].
- Eigenfunctions: $y_n(x) = \sin(\omega_n x)$.