Lecture 5.3: The transport and wave equations

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Motivation

Some common one-dimensional PDEs

We’ve seen the heat equation: $u_t = c^2 u_{xx}$. In this lecture, we will introduce the transport equation, from which we will derive the wave equation: $u_{tt} = c^2 u_{xx}$. 
Example 1

Consider the following PDE involving a function $u(x, t)$:

$$\frac{\partial u}{\partial t} - c \frac{\partial u}{\partial x} = 0.$$
Example 2

Consider the following PDE involving a function \( u(x, t) \):

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0.
\]
The wave equation

Example 3

Consider the following PDE involving a function $u(x, t)$:

$$
\left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) u = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0
$$
The three most common two-variable PDEs

Summary

Let \( u(x, t) \) be a function of position \( x \) and time \( t \). Then
- the heat equation is \( u_t = c^2 u_{xx} \),
- the wave equation is \( u_{tt} = c^2 u_{xx} \).

One more

Let \( u(x, y) \) be a function of position \( (x, y) \). Then
- Laplace's equation is \( u_{xx} + u_{yy} = 0 \).
Example 3

Solve the following B/IVP for the wave equation:

\[ u_{tt} = c^2 u_{xx}, \quad u(0, t) = u(L, t) = 0, \quad u(x, 0) = x(L - x), \quad u_t(x, 0) = 1. \]
Example 3 (cont.)

The general solution to the following BVP for the wave equation:

\[ u_{tt} = c^2 u_{xx}, \quad u(0, t) = u(L, t) = 0, \quad u(x, 0) = x(L - x), \quad u_t(x, 0) = 1. \]

is \( u(x, t) = \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{cn\pi t}{L} \right) + b_n \sin \left( \frac{cn\pi t}{L} \right) \right] \sin \left( \frac{n\pi x}{L} \right) \). Now, we’ll solve the remaining IVP.