Lecture 6.1: The heat and wave equations on the real line

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Overview

Broad goal

Solve the following initial value problem for heat equation on the real line:

$$u_t = c^2 u_{xx}, \qquad u(x,0) = h(x), \qquad -\infty < x < \infty, \qquad t > 0.$$

This is often called a Cauchy problem.

We will then do the same thing for the wave equation.

The process consists of two steps:

(1) First solve an easier IVP: when $u(x,0) = H(x)$, the Heavyside function.

(2) Construct a solution to the original IVP using the solution to (1).

Let's start right away with the IVP above.

Step 1

Solve the related IVP for the heat equation on the real line and $t > 0$:

$$v_t = c^2 v_{xx}, \qquad v(x,0) = H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Cauchy problem for the heat equation

Step 1

Solve the following IVP for the heat equation on the real line and $t > 0$:

$$v_t = c^2 v_{xx}, \qquad v(x,0) = v_0 \cdot H(x) = \begin{cases} v_0 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Let's say that distance is measured in meters and time in seconds. Then the units are

$$v_t : \ \frac{deg}{sec}, \qquad v_{xx} : \ \frac{deg}{m^2}, \qquad c^2 : \ \frac{m^2}{sec}, \qquad v(x,t) : \ deg, \qquad v_0 : \ deg.$$

This means the v/v_0 and $x/\sqrt{4c^2 t}$ are dimensionless quantities, and so we can express one as a function of the other:

$$\frac{v}{v_0} = f\left(\frac{x}{\sqrt{4c^2 t}}\right).$$

For simplicity, set $v_0 = 1$, and substitute

$$v = f(z), \qquad z = \frac{x}{\sqrt{4c^2 t}}.$$

We can use the chain rule to compute v_t and v_{xx}, and plug these back into the PDE above.

Cauchy problem for the heat equation

Step 1 (continued)

Solve the following initial value problem for the heat equation on the real line:

$$v_t = c^2 v_{xx}, \qquad v(x,0) = H(x), \qquad -\infty < x < \infty, \quad t > 0.$$

We let $v = f(z)$, where $z = \frac{x}{\sqrt{4c^2 t}}$ and found $v_t = -\frac{1}{2}\frac{x}{\sqrt{4c^2 t^3}} f'(z)$ and $v_{xx} = \frac{1}{4c^2 t} f''(z)$.

Cauchy problem for the heat equation

Step 1 (continued)

The general solution to the Cauchy problem for the heat equation on the real line

$$v_t = c^2 v_{xx}, \qquad v(x,0) = H(x), \qquad -\infty < x < \infty, \quad t > 0.$$

is $v(x,t) = C_1 \int_0^{x/\sqrt{4c^2 t}} e^{-r^2}\, dr + C_2$. Now we'll solve the IVP.

Cauchy problem for the heat equation

Step 2

Solve the original initial value problem for heat equation on the real line:

$$u_t = c^2 u_{xx}, \qquad u(x,0) = h(x), \qquad -\infty < x < \infty, \quad t > 0.$$

Remarks

- The function $v(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4c^2 t}} e^{-r^2}\, dr$ solves the heat equation.
- If v solves the heat equation, so does v_x.
- The function $G(x,t) := \frac{1}{\sqrt{4\pi c^2 t}} e^{-x^2/(4c^2 t)}$ is called the fundamental solution to the heat equation, or the heat kernel.
- The function $G(x-y,t)$ solves the heat equation, and represents an initial unit heat source at y.

Cauchy problem for the heat equation

Summary

The solution to the initial value problem for the heat equation on the real line,

$$u_t = c^2 u_{xx}, \qquad u(x,0) = h(x), \qquad -\infty < x < \infty, \;\; t > 0,$$

is $u(x,t) = \displaystyle\int_{-\infty}^{\infty} h(y) \frac{1}{\sqrt{4\pi c^2 t}} e^{-(x-y)^2/(4c^2 t)} \, dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} h(x - r\sqrt{4c^2 t}) \, dr.$

This second form is called the Poisson integral representation, which results from the substitution $r = \frac{x-y}{\sqrt{4c^2 t}}$.

Cauchy problem for the wave equation and D'Alembert's formula

Example

Solve the following initial value problem for the wave equation on the real line:

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad -\infty < x < \infty, \quad t > 0.$$

Recall that the general solution to $u_{tt} = c^2 u_{xx}$ is

$$u(x,t) = F(x - ct) + G(x + ct),$$

where F and G are arbitrary functions.