Lecture 7.2: Eigenvalues and eigenfunctions of the Laplacian

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Overview

The Laplacian is the differenital operator

$$\Delta = \nabla^2 := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

In the previous lecture, we found the kernel of this operator (for $n = 2$) under various boundary conditions.

This amounted to solving a PDE called Laplace's equation:

$$\Delta u = 0.$$

In this lecture, we will find the eigenvalues and eigenfunctions of Δ.

This amounts to solving a PDE called the Helmholtz equation:

$$\Delta u = -\lambda u.$$

This equation arises when solving the heat and wave equations in two dimensions.

Dirichlet boundary conditions

Example 1

Find the general solution to the following BVP for the Helmholtz equation

$$u_{xx} + u_{yy} = -\lambda u, \qquad u(0, y) = u(\pi, y) = u(x, 0) = u(x, \pi) = 0.$$

Neumann boundary conditions

Example 2

Find the general solution to the following BVP for the Helmholtz equation

$$u_{xx} + u_{yy} = -\lambda u, \qquad u_x(0, y) = u_x(\pi, y) = u_y(x, 0) = u_y(x, \pi) = 0.$$

Mixed boundary conditions

Example 3

Find the general solution to the following BVP for the Helmholtz equation

$$u_{xx} + u_{yy} = -\lambda u, \qquad u(0, y) = u_x(\pi, y) = u_y(x, 0) = u_y(x, \pi) = 0.$$