Lecture 7.4: The Laplacian in polar coordinates

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Goal

To solve the heat equation over a circular plate, or the wave equation over a circular drum, we need translate the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \partial_x^2 + \partial_y^2$$

into polar coordinates (r, θ), where $x = r\cos\theta$ and $y = r\sin\theta$.

First, let's write $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in polar coordinates.

Some messy calculations

The Laplacian is the sum of the following two differential operators:

$$\left(\frac{\partial}{\partial x}\right)^2 = \left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\right)^2, \qquad \left(\frac{\partial}{\partial y}\right)^2 = \left(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\right)^2.$$

Next goal

The Laplacian operator in polar coordinates is

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{1}{r}\partial_r + \partial_r^2 + \frac{1}{r^2}\partial_\theta^2.$$

Find the eigenvalues λ_{nm} (fundamental frequencies) and the eigenfunctions $f_{nm}(r,\theta)$ (fundamental nodes).

Naturally, this depends on the boundary conditions.

Clearly, in θ, the BCs have to be periodic: $f(r,\theta+2\pi) = f(r,\theta)$.

In r, the BCs can be:

- Dirichlet: $f(a,\theta) = 0$
- Neumann: $f_r(a,\theta) = 0$
- Mixed: $\alpha_1 f(a,\theta) + \alpha_2 f_r(a,\theta) = 0$.

We will only consider Dirichlet BCs conditions in this lecture.

Dirichlet boundary conditions

Example

Solve the following BVP for the Helmholtz equation in polar coordinates

$$\Delta f = f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = -\lambda f, \qquad f(1, \theta) = 0, \qquad f(r, \theta + 2\pi) = f(r, \theta).$$

Summary so far

To solve the heat equation over a circular plate, or the wave equation over a circular drum, we need to translate the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \partial_x^2 + \partial_y^2$$

into polar coordinates (r, θ), where $x = r\cos\theta$ and $y = r\sin\theta$. This becomes the operator

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{1}{r}\partial_r + \partial_r^2 + \frac{1}{r^2}\partial_\theta^2.$$

Its eigenvalues and eigenfunctions are

$$\lambda_{nm} = \omega_{nm}^2, \qquad f_{nm}(r,\theta) = \cos(n\theta)\, J_n(\omega_{nm} r), \qquad g_{nm}(r,\theta) = \sin(n\theta)\, J_n(\omega_{nm} r),$$

where ω_{nm} is the m^{th} positive root of $J_n(r)$, the Bessel function of the first kind of order n.

These functions form a basis for the solution space of Helmholtz's equation, $\Delta u = -\lambda u$. As such, every solution $h(r, \theta)$ under Dirichlet BCs can be written as

$$h(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm} \overbrace{\cos(n\theta)\, J_n(\omega_{nm} r)}^{f_{nm}(r,\theta)} + b_{nm} \overbrace{\sin(n\theta)\, J_n(\omega_{nm} r)}^{g_{nm}(r,\theta)}$$

$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\omega_{nm} r)\big[a_{nm}\cos(n\theta) + b_{nm}\sin(n\theta)\big].$$

Fourier-Bessel series, revisited

Every solution $h(r,\theta)$ to

$$\Delta u = -\lambda u, \qquad u(1,\theta) = 0, \qquad u(r,\theta+2\pi) = u(r,\theta)$$

can be written uniquely as

$$h(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm}\overbrace{\cos(n\theta)\,J_n(\omega_{nm}r)}^{f_{nm}(r,\theta)} + b_{nm}\overbrace{\sin(n\theta)\,J_n(\omega_{nm}r)}^{g_{nm}(r,\theta)}$$

$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\omega_{nm}r)\big[a_{nm}\cos(n\theta) + b_{nm}\sin(n\theta)\big].$$

This is called a Fourier-Bessel series. By orthogonality, and the identity

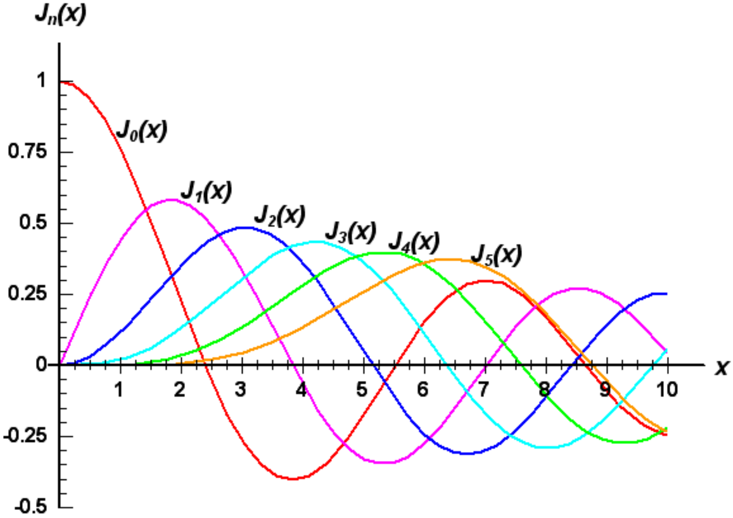
$$\big|\big|J_n(\omega x)\big|\big|^2 = \big\langle J_n(\omega x), J_n(\omega x)\big\rangle = \int_0^1 J_n^2(\omega x)x\,dx = \frac{1}{2}\big(J_{n+1}(\omega)\big)^2,$$

$$a_{nm} = \frac{\langle h, f_{nm}\rangle}{\langle f_{nm}, f_{nm}\rangle} = \frac{\iint_D h\cdot f_{nm}dA}{||f_{nm}||^2} = \frac{2}{J_{n+1}(\omega_{nm})^2}\int_{-\pi}^{\pi}\int_0^1 h(r,\theta)J_n(\omega_{nm}r)\cos(n\theta)\;r\,dr\,d\theta$$

$$b_{nm} = \frac{\langle h, g_{nm}\rangle}{\langle g_{nm}, g_{nm}\rangle} = \frac{\iint_D h\cdot g_{nm}dA}{||g_{nm}||^2} = \frac{2}{J_{n+1}(\omega_{nm})^2}\int_{-\pi}^{\pi}\int_0^1 h(r,\theta)J_n(\omega_{nm}r)\sin(n\theta)\;r\,dr\,d\theta.$$

Bessel functions of the first kind

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(\nu+m)!} \left(\frac{x}{2}\right)^{2m+\nu} .$$



Fourier-Bessel series from $J_0(x)$

$$f(x) = \sum_{n=0}^{\infty} c_n J_0(\omega_n x), \qquad J_0(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$$

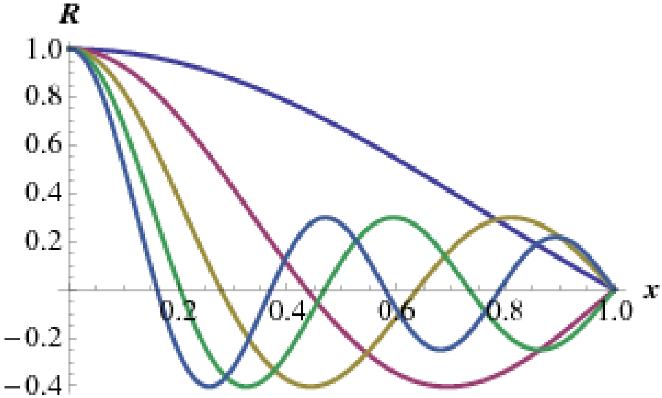
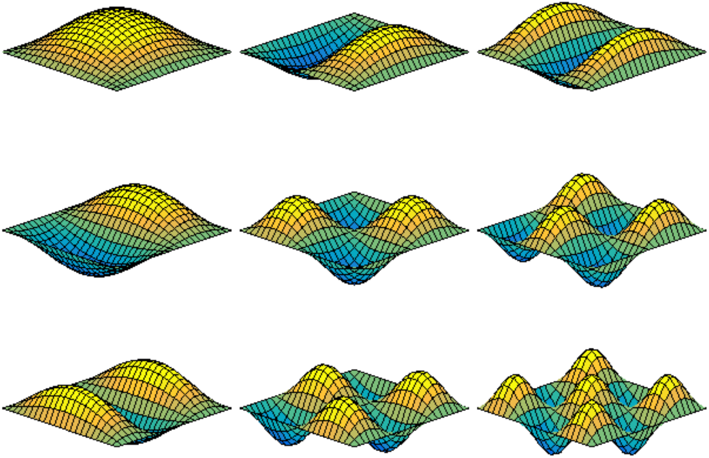


Figure: First 5 solutions to $(xy')' = -\lambda x^2$.

Eigenfunctions of the Laplacian in the unit square

$$\lambda_{nm} = n^2 + m^2, \qquad f_{nm}(x, y) = \sin nx \sin my$$



Eigenfunctions of the Laplacian in the unit disk

$$\lambda_{nm} = \omega_{nm}^2, \qquad f_{nm}(r,\theta) = \cos(n\theta)\, J_n(\omega_{nm} r), \qquad g_{nm}(r,\theta) = \sin(n\theta)\, J_n(\omega_{nm} r)$$