1. **(6 points)** `Library/UMass-Amherst/Abstract-Algebra/PS-Relations/Relations6.pg`

In this problem we work out step-by-step the procedure for checking an equivalence relation.

Denote by $\mathbb{Z}$ the set of all integers. Declare that two integers x, y are related if $x - y$ is an integer multiple of 5. In symbols:

$$x \sim y \Longleftrightarrow 5 \text{ divides } x - y.$$

We want to check if this is an equivalence relation. That means we need to check if $\sim$ is
(1) Reflexive
(2) Symmetric
(3) Transitive

We begin with (1). This means checking to make sure that for all integers x, we have $x \sim x$. Recall the definition of $\sim$ for this problem and we see that this is equivalent to saying that for all integers x, we have that $(x - x)$ is an integer multiple of 5.

Is this true? If so, enter Y; if not, enter an integer for which this is false. ____

Next, we check (2). This means checking to make sure that for all integers x, y, we have $x \sim y \Leftrightarrow y \sim x$. Unwind the definition of $\sim$ as we have done for (1) and we see that

$x \sim y \Longleftrightarrow$ ____ $= 5m$ for some integer m

$y \sim x \Longleftrightarrow$ ____ $= 5m$ for some integer m

Based on that, is (2) true? If so, enter Y; if not, enter a pair of integers for which this is false.
____

Finally, we check (3). This means checking to make sure that for all integers x, y, z, if $x \sim y$ and $y \sim z$ then $x \sim z$.

Is this true? If so, enter Y; if not, give a triple of integers for which this fails. ____

Finally, based on this calculation, is $\sim$ an equivalence relation on the set of integers? Enter Y or N. ____

2. **(6 points)** `Library/UMass-Amherst/Abstract-Algebra/PS-Relations/Relations2.pg`

For the following relations on the set of college students, determine if it satisfies each of the following conditions:

	Reflexive	Symmetric	Transitive	Equivalence Relation
$A \sim B \Leftrightarrow A$ is strictly taller than B	____	____	____	____
$A \sim B \Leftrightarrow A, B$ took 4 class(es) together	____	____	____	____
$A \sim B \Leftrightarrow A, B$ have the same major	____	____	____	____

Please enter Y or N in each of the boxes.

3. **(6 points)** `Library/UMass-Amherst/Abstract-Algebra/PS-Relations/Relations3.pg`

For the following relations on the set of POSITIVE integers, determine if it satisfies each of the following conditions:

	Reflexive	Symmetric	Transitive	Equivalence Relation
$m \sim n \Leftrightarrow 18$ divides $m-n$	_____	_____	_____	_____
$m \sim n \Leftrightarrow 16$ divides $m+n$	_____	_____	_____	_____
$m \sim n \Leftrightarrow 8$ divides mn	_____	_____	_____	_____

Please enter Y or N in each of the boxes.

4. **(6 points)** Library/UMass-Amherst/Abstract-Algebra/PS-Relations/Relations4.pg

For the following relations on the set of points on the plane, determine if it satisfies each of the following conditions (please enter Y or N in each of the boxes):

	Reflexive	Symmetric	Transitive	Equivalence Relation
$(u_1,u_2) \sim (w_1,w_2) \Leftrightarrow u_1 = w_1$	_____	_____	_____	_____
$(x_1,y_1) \sim (x_2,y_2) \Leftrightarrow$ either $(x_1,y_1) = (x_2,y_2)$ or the line segment joining the two points have a slope > 9	_____	_____	_____	_____
$(x_1,y_1) \sim (x_2,y_2) \Leftrightarrow$ the distance between the points is > 10	_____	_____	_____	_____

5. **(10 points)** Library/UMass-Amherst/Abstract-Algebra/PS-Relations/Relations1.pg

For each of the following relations on the set of real numbers, determine if it satisfies each the following conditions (enter Y or N in each of the boxes):

	Reflexive	Symmetric	Transitive	Equivalence Relation
$x \sim y \Leftrightarrow x^2 > y^2$	_____	_____	_____	_____
$x \sim y \Leftrightarrow x \leq y$	_____	_____	_____	_____
$x \sim y \Leftrightarrow x^2 + y^2 = 1$	_____	_____	_____	_____
$x \sim y \Leftrightarrow \lvert x-y \rvert < 7$	_____	_____	_____	_____
$x \sim y \Leftrightarrow xy = 0$	_____	_____	_____	_____

6. **(8 points)** Library/MC/Proofs/Relations/Equvalence01.pg

Order 7 of the following sentences so that they form a logical proof of the statement:

Suppose R is a symmetric and transitive relation on A (i.e. $A \times A$) . Further suppose that for each $a \in A$ that there exists $b \in A$ such that $(a,b) \in R$.

Show: R is an equivalence relation.

Quick Hint? **What makes R an equivalence relation?**

- R is reflexive
- Assume that R is symmetric and transitive on A and that each element in A is related to at least one other element in A.
- Let (a,b) be an arbitrary element of R.

- $(a,b) \in R$ and $(b,a) \in R$ implies $(a,a) \in R$ by transitivity
- R is an equivalence relation
- Too much may be the equivalent of none at all.
- $\exists b \in A$ such that $(a,b) \in R$
- Assume R is symmetric and transitive and $\forall a \in A, (a,a) \in R$.
- $(b,a) \in R$ by symmetry
- Let a be an arbitrary element of A.
- $(a,a) \in R \implies \exists b \in A$ such that $(a,b) \in R$ and $(b,a) \in R$ by transitivity

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7. **(8 points)** `Library/MC/Proofs/Relations/Equvalence02.pg`

Order 10 of the following sentences so that they form a logical proof of the statement:

For $A = Z \times Z$, define a relation R on A by:
$((a,b),(c,d)) \in R \iff ad = bc$
Prove that R is an equivalence relation on A.

- Consider $((a,b),(c,d)) \in R$.
- Then, $ad = bc$ and $cf = de$ and so $af = be$.
- Define R on $Z \times Z$ such that $((a,b),(c,d)) \in R \iff ad = bc$
- Then $ad = bc \implies bc = ad$ and so $(c,d)R(a,b)$.
- Hence, R is transitive. ¡br¿ For any $(a,b), ab = ba$.
- Hence, R is symmetric since $((a,b),(a,b)) \in R$.
- Nathan is a goob.
- Thus R is an equivalence relation.
- Hence, R is reflexive and $(a,b)R(c,d)$ means R is symmetric and transitive.
- $(a,b)R(c,d) \implies ab = cd$.
- Therefore $(a,b)R(a,b)$
- Hence R is symmetric. ¡br¿ Next consider $(a,b)R(c,d)$ and $(c,d)R(e,f)$
- Hence R is reflexive.
- $af = be \implies (a,b)R(e,f)$

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8. **(8 points)** `Library/MC/Proofs/Relations/Partition02.pg`
Among the options below there are 7 different partitions of the set $A =\ 0,1,2,...,15$. List them on the right according to the number of equivalence classes that each partition induces.

- 1,2,...14,15,16
- 1,2,...,5,6,8,...,14
- 0,1,2,...,8,9,10,...,15
- 0,15,1,14,2,13,3,12,...,7,8
- 0,1,2,...14,15
- S_0 =0,S_1 =3,6,9,S_2 =1,4,7,10,S_3 =2,5,8,11,$A - S_0 - S_1 - S_2 - S_3$
- $Z_0, Z_1, Z_2, Z_3, Z_4, Z_5$, ¡br¿ $Z_6, Z_7, ..., Z_{14}, Z_{15}$
- 0,1,2,...,15
- even positive integers less than 15,¡br¿odd positive integers less than 15,0,15
- even numbers less than 15,¡br¿odd numbers less than 15,15
- even positive integers less than 15,¡br¿odd positive integers less than 15,0,15

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9. **(6 points)** `Library/UMass-Amherst/Abstract-Algebra/PS-Relations/Relations5.pg`

Determine all pairs of integers A, B such that
$(m,n) \sim (u,v) \iff m - An = u - Bv$

is an equivalence relation on the set of all pairs of integers.

$A =$ __________
$B =$ __________

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10. **(6 points)** `Library/UMass-Amherst/Abstract-Algebra/PS-Functions/Functions1.pg`
Consider the function

$\varphi : \{8,9,...,16,17\} \to \{8,9,...,16,17\}$

x	8	9	10	11	12	13	14	15	16	17
$\varphi(x)$	13	16	8	9	14	17	12	11	15	9

(a) Is this one-to-one? _____ (Y/N)

(b) Is this onto? _____ (Y/N)

(c) Is this bijective? _____ (Y/N)

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11. **(5 points)** `Library/UMass-Amherst/Abstract-Algebra/PS-Functions/Functions2.pg`
Complete the following table of values of a function

$\varphi : \{6,7,...,14,15\} \to \{0,1,...,8,9\}$

x	6	7	8	9	10	11	12	13	14	15
$\varphi(x)$	4	___	___	0	3	8	___	___	5	___

so that φ is onto.

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12. **(6 points)** `Library/UMass-Amherst/Abstract-Algebra/PS-Relations/Relations7.pg`

Let X be the set $\{-18,-6,-1\}$. For the first three parts of this problem you are asked to define a function $f: X \to X$ so that the relation

$$u \sim w \Leftrightarrow w = f(u)$$

satisfies each of the following conditions.

(a) $\sim$ is reflexive

x	-18	-6	-1
$f(x)$	___	___	___

(b) $\sim$ is symmetric

x	-18	-6	-1
$f(x)$	___	___	___

(c) $\sim$ is transitive

x	-18	-6	-1
$f(x)$	___	___	___

(d) [optional: see your instructor] Let Y be an arbitrary non-empty set. Determine all functions $g: Y \to Y$ so that the relation

$$a \sim b \Leftrightarrow b = g(a)$$

is
- Reflexive
- Symmetric
- Transitive