Lecture 1.4: Binomial and multinomial coefficients

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Motivation

The number $\binom{n}{k}$ is called a binomial coefficient, and counts the number of k-element subsets of an n-element set.

The binomial coefficients satisfy a remarkable number of properties. In this lecture, we will explore these, and generalize them to the multinomial coefficients.

As a teaser, the entries in Pascal's triangle are actually binomial coefficients:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

A recursive identity for binomial coefficients

Theorem

The binomial coefficients satisfy the following recursive formula:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \qquad \text{for all } n > 0 \text{ and } 0 < k < n.$$

Proof 1 (algebraic)

Show that $\dfrac{n!}{k!(n-k)!} = \dfrac{(n-1)!}{(k-1)!(n-k)!} + \dfrac{(n-1)!}{k!(n-k-1)!} \dots$ □

Proof 2 (combinatorial)

Let's count, using two different methods, the number of ways to elect k candidates from a pool of n.

For the second method, assume that there is one "distinguished" candidate...

□

The binomial theorem

We will motivate the following theorem with an example:

$$\begin{aligned}(x+y)^6 &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6 \\ &= \tbinom{6}{0}x^6 + \tbinom{6}{1}x^5y + \tbinom{6}{2}x^4y^2 + \tbinom{6}{3}x^3y^3 + \tbinom{6}{4}x^2y^4 + \tbinom{6}{5}xy^5 + \tbinom{6}{6}y^6.\end{aligned}$$

Theorem

For any x, y and $n \geq 1$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof

Multiply out, or "FOIL" the product $\underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$.

This results in 2^n terms, all distinct length-n words in x and y. E.g., for $n = 6$:

$$xxxxxx + xxxxxy + \cdots + xyxyxy + \cdots + xxxyyy + \cdots + yyyyyy$$

There are $\binom{n}{k}$ words with exactly k instances of x, so this is the coefficient of $x^k y^{n-k}$.

The binomial theorem

Corollary

The n^{th} row of Pascal's triangle sums to $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof 1 (algebraic)

Take

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

and plug in $x = y = 1$. □

Proof 2 (combinatorial)

Let's enumerate the power set of $\{1, \dots, n\}$ of two different ways:

(i) Count the number of length-n binary strings

(ii) Count the number of size-k subsets, for $k = 0, 1, \dots, n$. □

A proof that establishes an identity by counting a carefully chosen set two different ways is called a combinatorial proof.

Multinomial coefficients

Exercise

A police department of 10 officers wants to have 5 patrol the streets, 2 doing paperwork, and 3 at the dohnut shop. How many ways can this be done?

Answer: $\binom{10}{5}\binom{5}{2}\binom{3}{3} = \frac{10!}{5!\,5!} \cdot \frac{5!}{2!\,3!} \cdot \frac{3!}{3!\,0!} = \frac{10!}{5!\,2!\,3!} = 2520.$

This is the same as counting the number of distinct permutations of the word

S S S S S P P D D D

Definition

Suppose that $n_1, \ldots, n_r$ are positive integers, and $n_1 + \cdots + n_r = n$. Then

$$\binom{n}{n_1, n_2, \ldots, n_r} := \frac{n!}{n_1!\; n_2! \;\cdots\; n_r!} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-\sum_{i=1}^{r-i} n_i}{n_r}$$

is called a multinomial coefficient. Binomial coefficients are the special case of $r = 2$.

Multinomials and words

Consider an alphabet with r letters: $\{s_1, \ldots, s_r\}$.

The number of length-n "words" (i.e., strings) that you can write using exactly n_i instances of s_i (where $n_1 + \cdots + n_r = n$) is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2!\, \cdots\, n_r!}.$$

Examples

(i) The number of distinct permutations of the letters in the word MISSISSIPPI is

$$\binom{11}{1, 4, 4, 2} = \frac{11!}{1!\,4!\,4!\,2!} = 34650.$$

(ii) How many length-13 strings can be made using 6 instances of $*$ ("star") and 7 instances of | ("bar")? Examples include:

$$*||***|||**|, \qquad ******|||||||, \qquad |*|*|*|*|*|*|.$$

Answer: $\binom{13}{6,7} = \frac{13!}{6!\,7!} = \binom{13}{6} = 1716.$

The multinomial theorem

Multinomial coefficients generalize binomial coefficients (the case when $r = 2$).

Not surprisingly, the Binomial Theorem generalizes to a Multinomial Theorem.

Theorem

For any $x_1, \ldots, x_r$ and $n > 1$,

$$(x_1 + \cdots + x_r)^n = \sum_{\substack{(n_1, \ldots, n_r) \\ n_1 + \cdots + n_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}.$$