Lecture 2.4: Axiomatic systems

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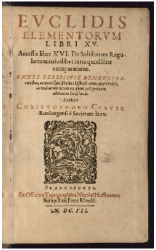
Some history



The ancient Greeks were the first people to study knowledge for the sake of knowledge.

Around 300 BC, Euclid wrote a series of thirteen books that he called The Elements.

It is a collection of definitions, axioms, and theorems on geometry, number theory, and "geometric algebra."

About 2200 years later, German mathematician David Hilbert axiomized geometry in a book *Foundations of Geometry*.

Euclid's axioms of geometry (c. 300 BC)

1. Any two points can be connected with a straight line.
2. Any line segment may be extended indefinitely.
3. A circle may be drawn with any given point as center and any given radius.
4. All right angles are equal.
5. For any given point not on a line, there is exactly one line through the point that does not intersect the line.

Some history

David Hilbert's axioms of geometry (1902 AD)

I. **Axioms of Incidence**

1. Any two points have a line that contains them.
2. Any two points have at most one line that contains them.
3. There are at least two points on a line, and at least three non-collinear points.
4. Any three non-collinear points have a plane containing them.
5. Any three non-collinear points have at most one plane containing them.
6. If two points on a line lie on a plane, then every point on the line lies on the plane.
7. If two planes have a point in common, then they have at least one more point in common.
8. There exist at least four points which do not lie on a plane.

II. **Axioms of Order**

1. If a point B lies between points A and C, then A, B, C are three distinct points of a line, and B lies between C and A.
2. Any two points A, B contain a point C on the line AB so that B lies between A and C.
3. Of any three points on a line, there is no more than one that lies between the other two.
4. Let A, B, C be collinear and ℓ a line on the plane they determine that does not meet these points. If ℓ passes through a point of the segment AB, it also passes through a point of AC or of BC.

Some history

David Hilbert's axioms of geometry (1902 AD)

III. **Axioms of Congruence**

1. If A, B are points on a line ℓ, and C is point on a line ℓ' (possibly the same), then there is a point D on a given side of ℓ' through C such that AB is congruent or equal to CD.
2. If two segments are congruent to a third segment, they are congruent to each other.
3. On a line ℓ, let AB and BC be segments that have no point other than B in common. On a line ℓ', let $A'B'$ and $B'C'$ be segments that have no point other than B' in common. If $AB = A'B'$ and $BC = B'C'$, then $AC = A'C'$.
4. Given an angle between two rays on a plane, a line ℓ and point A on another plane, and a designated side of ℓ, there is a unique ray through A that makes the same angle with ℓ.
5. "Side Angle Side" property for triangles.

IV. **Axiom of Parallels**

1. (Euclid) Given a point A not on a line ℓ, there is at most one line in the plane determined by A and ℓ that passes through A but does not intersect ℓ.

V. **Axioms of Continuity**

1. (Archimedes) Given two segments AB and CD, there exists a number n such that n segments CD, constructed contiguously from A, along the ray from A through B, will pass beyond B.
2. (Line completeness) Points cannot be added to lines while preserving the properties that follow from the prior axioms.

Axiomatic systems

Both Euclid's and Hilbert's constructions are examples of axiomatic systems.

Our goal for this lecture is to understand these, and how logic plays a role.

Understanding this is analogous to a writer understanding grammar, or a programmer learning how to write assembly code.

Definition

A axiomatic system consists of:

(i) A set or universe, U.

(ii) Definitions: sentences that explain the meaning of concepts that are related to U. These may use *undefined* terms.

(iii) Axioms: propositions that are self-evident. These must include *at least* the system of logic we've developed.

(iv) Theorems: true propositions derived from the axioms.

Axiomatic systems

Examples

1. Propositional calculus.
   - The universe consists of an infinite set of symbols: $\{0, 1, p, q, r, s, \dots\}$.
   - The definitions include truth tables, operations such as $\wedge$, $\neg$, etc.
   - One simple set of axioms, due to Jan Lukasiewicz:

$$\begin{array}{rcl} p & \to & (q \to p) \\ (p \to (q \to r)) & \to & ((p \to q) \to (p \to r)) \\ (\neg p \to \neg q) & \to & (q \to p) \end{array}$$

     (There are many other possibilities.)
   - Examples of theorems include DeMorgan's laws.
2. Euclidean geometry.
   - The universe consists of points, lines, circles.
   - The definitions include parallel lines, triangles, etc.
   - Euclid used five axioms (in addition to those of logic).
   - How to bisect an angle is a theorem.

Donald Knuth, one of the pioneers of computer science, said that writing a computer program from a set of specifications is comparable to writing a mathematical proof from a set of axioms.

Axiomatic systems

Theorems are usually expressed in terms of a finite number of propositions $p_1, \dots, p_n$, called the premises:

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n \Rightarrow C \qquad \text{"the conclusion"}$$

or just: $p_1, \dots, p_n$ imply C.

Definition

A proof of a theorem is a finite sequence of logically valid steps that demonstrates that the premise of a theorem imply its conclusion.

What constitutes a proof can be subjective, and depends on the audience.

The Four Color Theorem was famously proven in 1976 by massive computer assistance. Some "purists" did not consider that to be fully legitimate.

Propositional calculus is one of the few axiomatic systems for which any valid sentence can be determined to be T/F by *mechanical* means.

In theory:

- Every computer program is simply a long sequence of 0s and 1s.
- Every living being is simply a collection of atoms.
- All proofs can be done with truth tables.