Lecture 2.7: Quantifiers

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The existential quantifier

If $p(n)$ is a proposition over a universe U, its truth set T_p is a subset of U.

In many cases, such as when $p(n)$ is an equation, we are often concerned with two special cases:

- $T_p \neq \emptyset$: "$p(n)$ is true for some n,"
- $T_p = U$: "$p(n)$ is true for all n."

The existential quantifier

If $p(n)$ is a proposition over U with $T_p \neq \emptyset$, we say

"*there exists an $n \in U$ such that $p(n)$ (is true).*"

We write this as $(\exists n)_U(p(n))$.

The symbol $\exists$ is the existential quantifier. If the context is clear, we can just say $(\exists n)(p(n))$.

If $T_p = \emptyset$, i.e., if $(\exists n)(p(n))$ is *false*, then we can write $(\nexists n)_U(p(n))$.

"*there does not exist $n \in U$ such that $p(n)$ is true.*"

The existential quantifier

Examples

1. $(\exists k)_{\mathbb{Z}}(k^2 - k - 12 = 0)$ says that there is an integer solution to $k^2 - k - 12 = 0$.
2. $(\exists k)_{\mathbb{Z}}(3k = 102)$ says that 102 is a multiple of 3.
3. The statement $(\exists k)_{\mathbb{Z}}(3k = 100)$ is false, but $(\nexists k)_{\mathbb{Z}}(3k = 100)$ is true.
4. Since the solution set to $x^2 + 1 = 0$ is $\{i, -i\}$, we can say

$$(\nexists x)_{\mathbb{R}}(x^2 + 1 = 0), \qquad\qquad (\exists x)_{\mathbb{C}}(x^2 + 1 = 0).$$

The universal quantifier

Definition

If $p(n)$ is a proposition over U with $T_p = U$, we say

"*for all* $n \in U$, $p(n)$ *(is true)*"

We write this as $(\forall n)_U(p(n))$.

The symbol $\forall$ is the universal quantifier. If the context is clear, we can write $(\forall n)_U(p(n))$.

Unlike the symbol $\nexists$ for "there does not exist", the notation $\not\forall$ is not used. (Why?)

Examples

1. We can use a universal quantifier to say that the square of every real number is non-negative: $(\forall x)_{\mathbb{R}}(x^2 \geq 0)$.
2. $(\forall n)_{\mathbb{Z}}(n + 0 = 0 + n = n)$ is the identity property of zero for addition, over the integers.

Universal quantifier	Existential quantifier
$(\forall n)_U(p(n))$	$(\exists n)_U(p(n))$
$(\forall n \in U)(p(n))$	$(\exists n \in U)(p(n))$
$\forall n \in U,\ p(n)$	$\exists n \in U$ such that $p(n)$
$p(n)$, $\forall n \in U$	$p(n)$, for some $n \in U$
$p(n)$ is true for all $n \in U$	$p(n)$ is true for some $n \in U$

The negation of quantified propositions

Motivating example

Over the universe of animals, define

$$F(x): \quad x \text{ is a fish}, \qquad\qquad W(x): \quad x \text{ lives in water}.$$

The proposition $W(x) \to F(x)$ is not always true.

In other words: $(\forall x)(W(x) \to F(x))$ is false.

Equivalently, there exists an animal that lives in the water and is not a fish. That is,

$$\begin{aligned}\neg\Big((\forall x)(W(x) \to F(x))\Big) &\Leftrightarrow (\exists x)\Big(\neg(W(x) \to F(x))\Big) \\ &\Leftrightarrow (\exists x)(W(x) \wedge \neg F(x)).\end{aligned}$$

Big idea

The negation of a universally quantified proposition is an existentially quantified proposition:

$$\neg\Big((\forall n)_U(p(n))\Big) \quad \Leftrightarrow \quad (\exists n)_U(\neg p(n)).$$

The negation of an existentially quantified proposition is a universally quantified proposition:

$$\neg\Big((\exists n)_U(p(n))\Big) \quad \Leftrightarrow \quad (\forall n)_U(\neg p(n)).$$

The negation of quantified propositions

More examples

1. The ancient Greeks discovered that $\sqrt{2}$ is irrational. Two ways to state this symbolically are:
$$\neg\Big((\exists r)_{\mathbb{Q}}(r^2 = 2)\Big), \qquad \text{and} \qquad (\forall r)_{\mathbb{Q}}(r^2 \neq 2).$$

2. The following equivalent propositions are either both true or both false:
$$\neg\Big((\forall n)(n^2 - n + 41 \text{ is composite})\Big) \qquad \Leftrightarrow \qquad (\exists n)(n^2 - n + 41 \text{ is prime})\Big).$$

Multiple quantifiers (of one type)

Propositions with multiple variables can be quantified multiple times. For example, the proposition

$$p(x,y) : x^2 - y^2 = (x+y)(x-y)$$

is a tautology over the real numbers.

Here are three ways to write this with universal quantifiers:

$$(\forall (x,y))_{\mathbb{R}\times\mathbb{R}}(p(x,y)), \qquad (\forall x)_{\mathbb{R}}\Big((\forall y)_{\mathbb{R}}(p(x,y))\Big), \qquad (\forall y)_{\mathbb{R}}\Big((\forall x)_{\mathbb{R}}(p(x,y))\Big).$$

Consider the proposition over $\mathbb{R}\times\mathbb{R}$

$$q(x,y) : x - y = 1 \text{ and } y = x^2 - 1$$

which has solution set $T_q = \{(0,-1),(1,0)\}$.

Here are three ways to write this with universal quantifiers:

$$(\exists (x,y))_{\mathbb{R}\times\mathbb{R}}(q(x,y)), \qquad (\exists x)_{\mathbb{R}}\Big((\exists y)_{\mathbb{R}}(q(x,y))\Big), \qquad (\exists y)_{\mathbb{R}}\Big((\exists x)_{\mathbb{R}}(q(x,y))\Big).$$

Rule of thumb

Quantifiers of the same type can by arranged in any order without logically changing the meaning of the proposition.

Negating multiple quantifiers (of one type)

For another example, consider the following proposition which is always false:

$$p(x,y) : x+y=1 \text{ and } x+y=2.$$

We can express this us by negating a proposition involving existential quantifiers:

$$\begin{aligned}
\neg\Big((\exists x)_{\mathbb{R}}\big((\exists y)_{\mathbb{R}}(p(x,y))\big)\Big) &\Leftrightarrow \neg\Big((\exists y)_{\mathbb{R}}\big((\exists x)_{\mathbb{R}}(p(x,y))\big)\Big) \\
&\Leftrightarrow (\forall y)_{\mathbb{R}}\Big(\neg\big((\exists x)_{\mathbb{R}}(p(x,y))\big)\Big) \\
&\Leftrightarrow (\forall y)_{\mathbb{R}}\big((\forall x)_{\mathbb{R}}(\neg p(x,y))\big) \\
&\Leftrightarrow (\forall x)_{\mathbb{R}}\big((\forall y)_{\mathbb{R}}(\neg p(x,y))\big).
\end{aligned}$$

Multiple quantifiers (mixed)

When existential and universal quantifiers are mixed, the order cannot be changed without possibly logically changing the meaning.

For example, the following two propositions are different:

$$p : (\forall a)_{\mathbb{R}^+}\Big((\exists b)_{\mathbb{R}^+}(ab = 1)\Big), \qquad\qquad q : (\exists b)_{\mathbb{R}^+}\Big((\forall a)_{\mathbb{R}^+}(ab = 1)\Big).$$

Note that p is true, but q is false.

One way to see why q is false is to verify that $\neg q$ is true:

$$\begin{aligned} \neg\Big((\exists b)_{\mathbb{R}^+}\big((\forall a)_{\mathbb{R}^+}(ab = 1)\big)\Big) &\Leftrightarrow (\forall b)_{\mathbb{R}^+}\neg\Big((\forall a)_{\mathbb{R}^+}(ab = 1)\Big) \\ &\Leftrightarrow (\forall b)_{\mathbb{R}^+}\Big((\exists a)_{\mathbb{R}^+}(ab \neq 1)\Big). \end{aligned}$$

Sometimes, we get "lucky" and changing the order does not change the logical meaning, but that is rare. One example:

$$p : (\forall a)_{\mathbb{R}}\Big((\exists b)_{\mathbb{R}^+}(ab = 0)\Big), \qquad\qquad q : (\exists b)_{\mathbb{R}}\Big((\forall a)_{\mathbb{R}^+}(ab = 0)\Big).$$