Lecture 3.5: Rational and irrational numbers

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Overview

Definition

A real number r is rational if $r = \frac{a}{b}$ for integers $a, b \in \mathbb{Z}$. Otherwise, it is irrational.

Examples (not all with proof)

1. Every integer is rational, because $n = \frac{n}{1}$.
2. The sum of two rational numbers is rational.
3. Every decimal that terminates is rational. For example, $1.234 = 1 + \frac{234}{1000} = \frac{1234}{1000}$.
4. Every repeating decimal is rational. For example, if $x = 0.121212\ldots$, then

$$99x = 100x - x = 12.12121212\ldots - 0.12121212\ldots = 12,$$

   so $99x = 12$, i.e., $x = \frac{12}{99}$.
5. The numbers $\sqrt{2}$, π, and e are irrational.

Exercise

Show that every repeating decimal is rational.

Basic properties

Proposition

(i) If r and s are rational, then $r + s$ and rs are rational.

(ii) If $r \neq 0$ is rational and s is irrational, then $r + s$ and rs are irrational.

(iii) If r and s are irrational, then $r + s$ is ...???

Proof

Proofs of irrationality

Theorem (5^{th} century B.C.)

$\sqrt{2}$ is irrational.

Proof

Suppose for sake of contradiction that $\sqrt{2} = \dfrac{m}{n}$, for some integers m, n, with no common prime factors. This means that

$$2 = \frac{m^2}{n^2},$$

or equivalently, $2n^2 = m^2$.

How can we find a contradiction from this...?

Proofs of irrationality

Exercises

(i) Prove that $\sqrt{3}$ is irrational.

(ii) Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

(iii) Prove that $\sqrt[3]{2}$ is irrational.