Lecture 4.5: Cardinality and infinite sets

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4190, Discrete Mathematical Structures

Set cardinality

Question

What does it means for two sets X and Y to have the same size?

This is easy if the sets are finite. But what about the following sets:

- $2\mathbb{N}^+$ (positive even numbers)
- $\mathbb{N}^+$ (positive integers)
- $\mathbb{N}$ (non-negative integers)
- $\mathbb{Z}$ (integers)
- $\mathbb{Q}$ (rational numbers)
- $\mathbb{R}$ (real numbers)
- $\mathcal{F} := \{\text{functions } f\colon \mathbb{R} \to \mathbb{R}\}$

Clearly,

$$2\mathbb{N}^+ \subsetneq \mathbb{N}^+ \subsetneq \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathcal{F}$$

(assuming we associate the constant functions with real numbers).

But do any of these have the same size, and if so, what does that mean?

Recall some definitions

Definition

Let $f : X \to Y$ be a function. Then

- f is injective, or 1–1, if $f(x) = f(y)$ implies $x = y$.
- f is surjective, or onto, if $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$.
- f is bijective if it is both 1–1 and onto.

The notation $f : X \hookrightarrow Y$ means f is 1–1.

The notation $f : X \twoheadrightarrow Y$ means f is onto.

If $f : X \to Y$ is bijective, then there is a 1–1 correspondence between elements of X and Y.

When f is bijective, we can define its inverse function, $f^{-1} : Y \to X$.

Definition

Two sets X, Y have the same cardinality if there exists a bijection $f : X \to Y$.

Some "problems" with infinity

What do you think the following equations "should be"?

- $1 + \infty =$
- $^1/_\infty =$
- $^\infty/_1 =$
- $^0/_\infty =$
- $2 \cdot \infty =$
- $\infty \cdot \infty =$
- $\infty - \infty =$
- $\infty - \frac{1}{4}\infty =$

Let's consider the following thought experiment.

Suppose Farmer A plants a seed every day, but every fourth day, a bird comes along and eats the seed he just planted.

• • • • • • • • • • • • ⋯

Suppose Farmer B plants a seed every day, but every fourth day, a bird comes along and eats the first seed he planted.

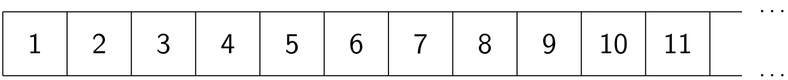
• • • • • • • • • • • • ⋯

Which farmer has more plants remaining "at the end of time"?

Hilbert's Hotel

Here's another thought experiment, proposed by David Hilbert in 1924.

Imagine a hotel that has infinitely rooms, but no vacancies. However, the manager is able to shuffle people around to open up a room, if needed.



If the hotel is full, what can the manager do to accommodate:

- A single person who shows up wanting a room?
- 10 people who show up wanting rooms?
- An "infinite football team" that shows up wanting rooms?
- A second "infinite football team" that shows up wanting room?
- The "rational number football team" that shows up wanting rooms?
- The "real number football team" that shows up wanting rooms?

Cardinality of the rationals

Suppose a bus containing the "positive rational number football team" shows up to Hilbert's hotel, which is empty.

How could the manager assign room numbers?

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
$5/1$	$5/2$	$5/3$	$5/4$	$5/5$	$5/6$	$\cdots$
$4/1$	$4/2$	$4/3$	$4/4$	$4/5$	$4/6$	$\cdots$
$3/1$	$3/2$	$3/3$	$3/4$	$3/5$	$3/6$	$\cdots$
$2/1$	$2/2$	$2/3$	$2/4$	$2/5$	$2/6$	$\cdots$
$1/1$	$1/2$	$1/3$	$1/4$	$1/5$	$1/6$	$\cdots$

1	2	3	4	5	6	7	8	9	10	11	$\cdots$

Cantor's diagonal argument

Theorem (Georg Cantor, 1891)

$|\mathbb{R}| > |\mathbb{Q}|$.

Proof

It suffices to show that $|[0, 1)| > |\mathbb{N}|$.

For sake of contradiction, suppose that there was a bijection $f : \mathbb{N} \to [0, 1)$.

Let's make a table of the numbers $f(0)$, $f(1)$, $f(2)$, $f(3)$, ...

There are infinitely many infinities

Theorem

For any set A, we have $\left|2^A\right| > |A|$.

Proof

It suffices to show that there is no surjection $f\colon A \to 2^A$.

Consider a function $f\colon A \to 2^A$, and define

$$D = \left\{a \in A \mid a \notin f(a)\right\} \in 2^A.$$

Take any $a \in A$. We will show that $f(a) \neq D$, and so f is not onto.

Case 1. If $a \in D$, then by definition, $a \notin f(a)$.

This means that $f(a) \neq D$, because D contains a but $f(a)$ doesn't.

Case 2. If $a \notin D$, then by definition, $a \in f(a)$.

But this means that $f(a) \neq D$, because $f(a)$ contains a but D doesn't. □

More fun facts

Definition

Define $\aleph_0 = |\mathbb{N}|$. A set S such that $|S| = \aleph_0$ is said to be countably infinite. The term countable (usually) means at most countably infinite.

If $|S| > \aleph_0$, then S is uncountable.

- The rational numbers can be "covered" with intervals whose total length is 1.
- The set of real-valued functions is strictly larger than $\mathbb{R}$. The latter's cardinality is called the continuum, denoted c.
- To answer our question from the beginning of the lecture:

$$|2\mathbb{N}^+| = |\mathbb{N}^+| = |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}| < |\mathcal{F}|.$$

- The question of whether there exists a set S with $\aleph_0 < |S| < c$ is called the continuum hypothesis.
- Results from Gödel and Paul Cohen have showed that the continuum hypothesis is undecidable – it lies outside of the standard axioms of set theory!