

## TOPICS: VECTOR SPACES, LINEAR INDEPENDENCE, AND BASES

1. For each of the following sets, determine if it is a *vector space* over  $\mathbb{R}$ . If it is, give an explicit *basis* and compute its *dimension*. If it isn't, explain why not by giving an example of how one of the vector space properties fails.
  - (a) The set of points in  $\mathbb{R}^3$  with  $x = 0$ .
  - (b) The set of points in  $\mathbb{R}^2$  with  $x = y$ .
  - (c) The set of points in  $\mathbb{R}^3$  with  $x = y$ .
  - (d) The set of points in  $\mathbb{R}^3$  with  $z \geq 0$ .
  - (e) The plane in  $\mathbb{R}^3$  defined by the equation  $z = 2x - 3y$ .
  - (f) The set of unit vectors in  $\mathbb{R}^2$ .
  - (g) The set of polynomials of degree  $n$ .
  - (h) The set  $\mathbb{R}_n[x]$  of polynomials of degree at most  $n$ .
  - (i) The set of polynomials of degree at most  $n$ , with only even-powers of  $x$ .
  - (j) The set  $\text{Per}_{2\pi}(\mathbb{R})$  of piecewise continuous functions, i.e.,  $f$  such that  $f(x) = f(x+2\pi)$ .
  - (k)  $\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R}\}$ .
  
2. Let  $v_1, v_2, w$  be three *linearly independent* vectors in  $\mathbb{R}^3$ . That is, they do not all lie on the same plane. For each of the following (infinite) set of vectors, carefully sketch it in  $\mathbb{R}^3$ , and determine whether or not it is a vector space (i.e., a *subspace* of  $\mathbb{R}^3$ ). Explain your reasoning.
  - (a)  $\{Cv_1 \mid C \in \mathbb{R}\}$
  - (b)  $\{Cv_1 + w \mid C \in \mathbb{R}\}$
  - (c)  $\{C_1v_1 + C_2v_2 \mid C_1, C_2 \in \mathbb{R}\}$
  - (d)  $\{C_1v_1 + C_2v_2 + w \mid C_1, C_2 \in \mathbb{R}\}$
  
3. Find the general solution to each of the following ODEs. Then, decide whether or not the set of solutions form a vector space. Explain your reasoning. Compare your answers to the previous problem. Recall that the general solution has the form  $y(t) = y_h(t) + y_p(t)$ .
  - (a)  $y' - 2y = 0$
  - (b)  $y' - 2y = 1$
  - (c)  $y'' + 4y = 0$
  - (d)  $y'' + 4y = e^{3t}$
  
4. For each of the following pairs of vectors  $v_1 = (x_1, y_1)$  and  $v_2 = (x_2, y_2)$  in (a)–(e), carry out the following steps:
  - (i) The lines through  $v_1$  and  $v_2$  generate a grid (of parallelograms) on the  $xy$ -plane. Sketch  $v_1, v_2$ , and this grid.
  - (ii) Find the area of one of the parallelograms by computing the determinant of the matrix  $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$ . Is this matrix invertible?
  - (iii) Determine whether  $\{v_1, v_2\}$  is a basis of  $\mathbb{R}^2$ .

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| (a) $v_1 = (1, 0), v_2 = (0, 1)$  | (d) $v_1 = (1, 1), v_2 = (1, 2)$   |
| (b) $v_1 = (2, 0), v_2 = (0, 2)$  | (e) $v_1 = (1, 2), v_2 = (1, 1)$   |
| (c) $v_1 = (\frac{1}{2}, \frac{1}{2}), v_2 = (\frac{1}{2}, -\frac{1}{2})$ | (f) $v_1 = (2, -1), v_2 = (-4, 2)$ |

Summarize your conclusions in a sentence or two.

5. For each of the following triples of vectors  $v_1 = (x_1, y_1, z_1)$ ,  $v_2 = (x_2, y_2, z_2)$ , and  $v_3 = (x_3, y_3, z_3)$ , carry out the following steps:

- (i) Sketch  $v_1, v_2$ , and  $v_3$  in  $\mathbb{R}^3$ .

- (ii) Use a computer to calculate the determinant of  $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ . Is it invertible?

- (iii) The lines through  $v_1, v_2$ , and  $v_3$  generate a tessellation (of parallelepipeds) in  $\mathbb{R}^3$ . What do you think the volume of each parallelepiped is?

- (iv) Describe in words (e.g., line, plane, all of  $\mathbb{R}^3$ ) the subspace  $\text{Span}\{v_1, v_2, v_3\}$ . Is  $\{v_1, v_2, v_3\}$  a basis of  $\mathbb{R}^3$ ?

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| (a) $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1)$  |
| (b) $v_1 = (2, 0, 0), v_2 = (0, 2, 0), v_3 = (0, 0, 2)$  |
| (c) $v_1 = (1, 0, 0), v_2 = (0, 1, 1), v_3 = (3, 1, 1)$  |
| (d) $v_1 = (1, 0, 0), v_2 = (0, 2, -1), v_3 = (1, 1, 1)$ |

Summarize your conclusions in a sentence or two.

6. For each of the following, a vector space  $V$  is given, along with a finite set  $S \subset V$ . Denote the subspace of  $V$  spanned by  $S$  as  $\text{Span}(S)$ . Find an explicit basis for  $\text{Span}(S)$  and compute its dimension.

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| (a) $V = \mathbb{R}^3, S = \{(1, 0, 0), (0, 1, 1), (1, 1, 1), (3, 1, 1)\}$ .                              |
| (b) $V = \mathbb{R}^2, S = \{(1, 0), (0, 1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})\}$ . |
| (c) $V = \mathcal{C}^\infty(\mathbb{R}), S = \{e^{3x}, e^{-3x}, \cosh 3x, \sinh 3x\}$ .                   |