Topics: The Wronskian, affine spaces, and linear ODEs

1. Solve the following ODEs:

   (a) $y' + 9y = 0$
   (b) $y'' + 9y = 0$
   (c) $y'' + 5y' + 4y = 0$
   (d) $4y'' + 4y' + 5y = 0$
   (e) $y' + 9y = 18$
   (f) $y'' + 9y = e^{3t}$
   (g) $y'' + 5y' + 4y = e^t$
   (h) $4y'' + 4y' + 5y = e^{2t}$

2. The function $y_1(t) = t$ is a solution of $t^2y'' + 5ty' - 5y = 0$.

   (a) Use the *reduction of order* technique to find a second solution, $y_2(t)$.
   (b) Compute the Wronskian of $y_1(t)$ and $y_2(t)$.

3. Let v_1 and v_2 be two linearly independent vectors in $\mathbb{R}^3$, i.e., they determine a plane P.

   (a) Sketch the plane $P = \{C_1v_1 + C_2v_2 \mid C_1, C_2 \in \mathbb{R}\}$ in $\mathbb{R}^3$.
   (b) Sketch the vectors $w_1 := \frac{1}{2}v_1 + \frac{1}{2}v_2$ and $w_2 := \frac{1}{2}v_1 - \frac{1}{2}v_2$. Do these "determine" the same plane? In other words, is the following (infinite) set of vectors

   $$\left\{C_1\left(\tfrac{1}{2}v_1 + \tfrac{1}{2}v_2\right) + C_2\left(\tfrac{1}{2}v_1 - \tfrac{1}{2}v_2\right) \mid C_1, C_2 \in \mathbb{R}\right\}$$

   the same as $\{C_1v_1 + C_2v_2 \mid C_1, C_2 \in \mathbb{R}\}$?
   (c) Is $\{v_1, v_2\}$ a basis for P? Is $\{w_1, w_2\}$ a basis for P? Why or why not?
   (d) Calculate the determinant of the matrix $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$. Is this matrix invertible? How does this relate to HW 1, Problem #4?

4. In this problem, we will understand from a linear algebra perspective why we can write solutions to ODEs for simple harmonic motion using complex exponentials or sines and cosines.

   (a) Consider the ODE $y'' - \omega^2 y = 0$. If we assume that $y(t) = e^{rt}$ and plug this back in and solve for r, we get that $r = \pm\omega$. Therefore, the general solution is $y(t) = C_1e^{\omega t} + C_2e^{-\omega t}$, i.e., $\{e^{\omega t}, e^{-\omega t}\}$ is a *basis* for the solution space. Using the fact that $e^{\omega t} = \cosh \omega t + \sinh \omega t$, find a basis for the solution space involving hyperbolic trig functions, and write the general solution using these functions. *Hint*: Look at the previous problem!
   (b) Repeat the above exercise for the ODE $y'' + \omega^2 y = 0$. Specifically, if we assume that $y(t) = e^{rt}$, then we get that $r = \pm i\omega$. Therefore, the general solution is $y(t) = C_1e^{i\omega t} + C_2e^{-i\omega t}$, i.e., $\{e^{i\omega t}, e^{-i\omega t}\}$ is a basis for the solution space. Using Euler's formula: $e^{i\omega t} = \cos \omega t + i \sin \omega t$, find a basis for the solution space involving sine and cosines, and write the general solution using these functions.

5. Suppose that the temperature of glass of water sitting outside obeys Newton's law of cooling. It is reasonable to expect that the ambient temperature is not constant, but rather a function $A(t)$ of time. Moreover, it is reasonable to assume that the ambient temperature is sinusoidal, so let $A(t) = B\sin\omega t$, where B is some constant. We now have the following equation that we wish to solve:

$$T' = k(B\sin\omega t - T)\,.$$

   (a) Solve the related homogeneous equation, $T_h' = -kT_h$.

   (b) To solve the ODE, we need to find any particular solution, so assume there is one of the form $T_p(t) = a\cos\omega t + b\sin\omega t$. Plug this back into the ODE, equate coefficients of the sine and cosine terms, and solve for a and b in terms of the amplitude B, the frequency ω, and the constant k.

   (c) Find the general solution to this ODE.

   (d) Give a qualitative physical description of what the particular solution $T_p(t)$ represents, and why. [Hint: Consider the long-term behavior of the temperature $T(t)$.]