TOPICS: THE FROBENIUS METHOD AND BESSEL'S EQUATION

1. The differential equation $(1-x^2)y'' - xy' + p^2y = 0$, where p is a constant, is known as Chebyshev's equation. It can be rewritten in the form

$$y'' + P(x)y' + Q(x)y = 0$$
, $P(x) = -\frac{x}{1 - x^2}$, $Q(x) = \frac{p^2}{1 - x^2}$.

- (a) If P(x) and Q(x) are represented as a power series about $x_0 = 0$, what is the radius of convergence of these power series?
- (b) Assume that the general solution has the form $\sum_{n=0}^{\infty} a_n x^n$, and find a recurrence for a_{n+2} in terms of a_n . [Hint: Before plugging back in, multiply through by $1-x^2$.]
- (c) Use the recurrence to determine a_n in terms of a_0 and a_1 , for $2 \le n \le 9$.
- (d) For each $p \in \mathbb{N}$, there is a unique polynomial solution $T_p(x)$ known as the Chebyshev polynomial of degree p. Find $T_3(x)$.
- 2. For each of the following ODEs, determine whether x=0 is an ordinary or singular point. If it is singular, determine whether it is regular or not. (Remember, first write each ODE in the form y'' + P(x)y' + Q(x)y = 0.)

(a)
$$y'' + xy' + (1 - x^2)y = 0$$

(c)
$$x^2y'' + 2xy' + (\cos x)y = 0$$

(b)
$$y'' + (1/x)y' + (1 - (1/x^2))y = 0.$$
 (d) $x^3y'' + 2xy' + (\cos x)y = 0.$

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- 3. Consider the differential equation 3xy'' + y' + y = 0. Since $x_0 = 0$ is a regular singular point, there is a generalized power series solution of the form $y(x) = \sum_{n} a_n x^{n+r}$.
 - (a) Determine the indicial equation (solve for r) and the recurrence relation for the coefficients.
 - (b) Find two linearly independent generalized power series solutions (i.e., a basis for the solution space).
 - (c) Determine the radius of convergence of each of these solutions. [Hint: First compute the radius of convergence of xP(x) and $x^2Q(x)$ and apply Frobenius].
- 4. Consider the differential equation 2xy'' + y' + xy = 0. Since $x_0 = 0$ is a regular singular point, there is a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.
 - (a) Determine the indicial equation (solve for r) and the recursion formula.
 - (b) Find a basis for the solution space and use this to write the general solution.
 - (c) What is the radius of convergence of each of these two linearly independent solutions?

- 5. Consider the differential equation xy'' + 2y' xy = 0.
 - (a) Show that x = 0 is a regular singular point.
 - (b) Show that if $a_0 = 0$, then r = -1 is one solution for the indicial equation.
 - (c) For r = -1 and $a_0 = 0$, find the recurrence relation for a_{n+2} in terms of a_n .
 - (d) Still assuming that $a_0 = 0$, write the solution from (b) as a generalized power series.
 - (e) Write this solution using a standard hyperbolic trigonometric function.
- 6. Consider the ODE $y'' + e^{-x}y = 0$. Change variables by setting $t = 2e^{-x/2}$, which will reduce this ODE to a Bessel's equation. Using the known solution to Bessel's equation, back-substitute to determine the solution y(x) to the original ODE.