TOPICS: REAL FOURIER SERIES, AND FOURIER SINE & COSINE SERIES

1. Find the Fourier series of the following functions *without* computing any integrals.

   (a) $f(x) = 2 - 3\sin 4x + 5\cos 6x$,

   (b) $f(x) = \sin^2 x$. [*Hint*: Use a standard trig identity.]

2. Consider the sawtooth wave defined on $[-1, 1]$ by the function $f(t) = t$, and extended to be periodic of period $T = 2$.

   (a) Sketch the graph of $f(t)$ on $[-7, 7]$.

   (b) Compute the Fourier series of $f(t)$.

   (c) The differential equation
   $$x''(t) + \omega^2 x(t) = f(t)$$
   describes the motion of a simple harmonic oscillator, subject to a driving force given by the sawtooth wave $f(t)$. Find the general solution by first solving the homogeneous equation, and then looking for a particular solution of the form
   $$x_p(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t)\,.$$

3. Consider the $2\pi$-periodic function defined on $[-\pi, \pi]$ by
   $$f(t) = \begin{cases} 0 & -\pi \le t < 0, \\ t & 0 \le t \le \pi, \end{cases}$$

   (a) Sketch the graph of $f(t)$ on $[-7\pi, 7\pi]$.

   (b) Compute the Fourier series of $f(t)$.

   (c) Sketch the graph of the resulting Fourier series. [It will be the same as the answer to Part (a) *except* at the points of discontinuity.]

   (d) Solve the differential equation $x''(t) + \omega^2 x(t) = f(t)$. Look for a particular solution of the form
   $$x_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt\,.$$

4. Determine which of the following functions are even, which are odd, and which are neither.

   (a) $f(x) = x^3 + 3x$

   (b) $f(x) = 4\sin 2x$

   (c) $f(x) = x^2 + |x|$

   (d) $f(x) = e^x$

   (e) $f(x) = \frac{1}{x}$

   (f) $f(x) = \frac{1}{2}(e^x + e^{-x})$

   (g) $f(x) = x\cos x$

   (h) $f(x) = \frac{1}{2}(e^x - e^{-x})$.

5. In this problem, we will investigate why in many Fourier series, every other coefficient is zero. This has to do with certain symmetries in the graph.
   (a) The Fourier series of an odd function consists only of sine-terms. What additional symmetry conditions on $f$ will imply that the sine coefficients with even indices will be zero (i.e., each $b_{2n} = 0$)? Give an example of a non-zero function satisfying this additional condition.
   (b) What symmetry conditions on $f$ will imply that the sine coefficients with odd indices will be zero (i.e., each $b_{2n+1} = 0$)? Give an example of a non-zero function satisfying this additional condition.
   (c) Sketch the graph of a non-zero even function, such that $a_{2n} = 0$ for all $n$.
   (d) Sketch the graph of a non-zero even function, such that $a_{2n+1} = 0$ for all $n$.

6. Consider the function $f(x) = x^2$ defined on the interval $[0, L]$. For this problem, you will determine the Fourier series, Fourier cosine series, and Fourier sine series of $f(x)$. Feel free to use a computer to find any indefinite integrals that you need.
   (a) Sketch the even extension of $f$ and compute its Fourier cosine series.
   (b) Sketch the odd extension of $f$ and compute its Fourier sine series.
   (c) Sketch the periodic extension of $f$ and compute its Fourier series.