Topics: Complex Fourier series, Fourier transforms, and Parseval's theorem

1. Consider the function $f(x) = L - x$ defined on the interval $[-L, L]$ and extended to be $2L$-periodic.

   (a) Sketch $f(x)$ on the interval $[-7L, 7L]$.

   (b) Compute the complex Fourier series of $f(x)$.

   (c) Find the real Fourier series of $f(x)$. [*Hint*: Use $a_n = c_n + c_{-n}$, and $b_n = i(c_n - c_{-n})$.]

   (d) Sketch the Fourier series of $f(x)$. [It will be the same as the answer to Part (a) *except* at the points of discontinuity.]

2. Consider the 2π-periodic function defined on $[-\pi, \pi]$ by

$$f(t) = \begin{cases} 0 & -\pi \leq t < 0, \\ t & 0 \leq t \leq \pi, \end{cases}$$

   (a) Sketch the graph of $f(t)$ on $[-7\pi, 7\pi]$.

   (b) Compute the complex Fourier series of $f(t)$.

   (c) Sketch the graph of the resulting Fourier series. [It will be the same as the answer to Part (a) *except* at the points of discontinuity.]

   (d) Solve the differential equation $x''(t) + \omega^2 x(t) = f(t)$. Look for a particular solution of the form

$$x_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}.$$

3. Find the Fourier transform of the function $f(x) = \begin{cases} e^{-ax} & x > 0 \\ 0 & x \leq 0 \end{cases}$

4. Consider the function defined by

$$f(x) = x^2 \qquad \text{for } -\pi < x \leq \pi.$$

   and extended to be periodic of period $T = 2\pi$.

   (a) Find the complex form Fourier series of $f(x)$. Feel free to use a computer to find the indefinite integral $\int x^2 e^{-inx}\, dx$.

   (b) Find the real form of the Fourier series.

   (c) Use Part (b), along with the real version of Parseval's identity to compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

5. Consider the 2π-periodic function f defined on $[-\pi, \pi]$ by $f(x) = |x|$.
   (a) What is $f(\pi)$?
   (b) Compute the Fourier series of f.
   (c) Plug $x = \pi$ into the Fourier series and use this to compute $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

6. Consider a complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L},$$

Prove the complex version of *Parseval's identity*, which says that

$$\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2 .$$

Observe that this is in a sense an infinite-dimensional version of the Pythagorean theorem, because

$$||f||^2 := \langle f, f \rangle := \frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2 := \sum_{n=-\infty}^{\infty} \left| \langle f, e^{in\pi x/L} \rangle \right|^2 .$$