Topics: Self-adjoint operators and Sturm-Liouville theory

For consistency, we will say that a *Sturm-Liouville equation* is a second-order differential equation in the following *self-adjoint form*:

$$-\frac{d}{dx}(p(x)y') + q(x)y = \lambda w(x)y\,, \tag{1}$$

where $p(x) > 0$ and $w(x) > 0$ is called the *weight*, or *density* function. If we divide through by $w(x)$, we can write this equation as $Ly = \lambda y$, where L is a *self-adjoint* linear operator. The possible values of λ are the *eigenvalues*, and solutions are the *eigenfunctions*.

1. Write the following differential equations in self-adjoint form. That is, put them in the above form, and find $p(x)$, $q(x)$, and the *weight* $w(x)$. Also, write out the corresponding linear operator L.
   (a) Airy's equation: $y'' + (\lambda - x)y = 0$
   (b) Laguerre's equation: $xy'' + (1-x)y' + \lambda y = 0$
   (c) An arbitrary linear equation: $y'' + P(x)y' + Q(x)y = \lambda R(x)y$. [*Hint*: Multiply through by an *integrating factor*, $e^{\int P(x)dx}$.]

2. In this problem, we will find all solutions to the Sturm-Liouville problem
$$-y'' = \lambda y, \qquad y'(0) = y(L) = 0.$$
   (a) First, suppose that $\lambda = 0$. That is, solve $y'' = 0$, $y'(0) = y(L) = 0$.
   (b) Next, suppose $\lambda = -\omega^2 \le 0$. That is, solve the boundary value problem $y'' = \omega^2 y$, $y'(0) = y(L) = 0$. [*Hint*: When the domain is finite, e.g., $[0, L]$, it is usually more convenient to use cosh and sinh instead of exponentials.]
   (c) Finally, suppose $\lambda = \omega^2 > 0$. That is, solve $y'' = -\omega^2 y$, $y'(0) = y(L) = 0$.
   (d) Summarize the results from Parts (a)–(c) in terms of the eigenvalues and corresponding eigenfunctions of a particular linear differential operator L. What is L?
   (e) Sketch the first four eigenfunctions on $[0, L]$.

3. By the main theorem of Sturm-Liouville theory, if we define an inner product as
$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}w(x)\,dx\,, \tag{2}$$
then the eigenfunctions $\{y_n(x)\}$ form an *orthogonal basis* (Note: not necessarily *orthonormal*!) for the space of functions, integrable on $[a, b]$ with $\langle f, f\rangle < \infty$ that satisfy the boundary condtions. This means that for any $f \in \mathsf{L}^2([a, b], w)$ with the same boundary conditions, we can write
$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)\,.$$

(a) Consider the Sturm-Liouville problem from Part (c) of the previous problem:

$$-y'' = \lambda y\,, \qquad y'(0) = 0\,, \quad y(L) = 0\,.$$

What is $w(x)$?

(b) The function $f(x) = x^2 - L^2$ is clearly continuous and satisfies $f'(0) = f(L) = 0$. Compute the *norm* $||f|| := \langle f, f\rangle^{1/2}$ of f.

(c) Since the eigenfunctions form a basis for the subspace of $\mathsf{L}^2([0, L]; w)$ that satisfy the above boundary conditions, we can write

$$x^2 - L^2 = \sum_{n=1}^{\infty} c_n y_n(x)\,, \qquad 0 \le x \le L\,.$$

Write down a formula for the c_n's. Leave your answer in terms of an integral – no need to actually compute it! [*Hint*: Don't forget that $y_n(x)$ isn't necessarily of unit length!]

4. Consider the following Sturm-Liouville problem:

$$-y'' - y' = \lambda y\,, \qquad y(0) = 0 \quad y(2) = 0\,.$$

(a) Find the eigenvalues and eigenfunctions. [*Hint*: You will encounter a *discriminant* of $D = 1 - 4\lambda$. As before, there will be three cases: $D = 0$, $D > 0$, and $D < 0$.]

(b) Write this differential equation in standard form, as in Eq. (1). [*Hint*: First, multiply through by an *integrating factor*, e^x.]

(c) Write a formula for $\langle y_n, y_m\rangle$ in terms of an integral. What is this integral equal to when $n \ne m$?

5. Consider the following Sturm-Liouville equation on $[-1, 1]$, called *Legendre's differential equation*:

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \tag{3}$$

In this problem, you will find the eigenvalue and eigenfunctions, which have already come up several times in this class in different settings.

(a) Write Legendre's equation into *self-adjoint form*, as in Equation 1. That is, find $p(x)$, $q(x)$, and $w(x)$, and the self-adjoint operator L. This is called a *singular Sturm-Liouville problem* on the inteveral $[a, b] = [-1, 1]$ because the function $p(x)$ satisfies $p(-1) = p(1) = 0$, and so boundary conditions on $y(x)$ are not needed.

(b) Assume that there is a power series solution of the form $\displaystyle\sum_{n=0}^{\infty} a_n x^n$. Plug this back into Eq. (3) and find the recurrence relation for the coefficients.

(c) Recall from HW 5 that a generalized power series solution will have radius of convergence $R = 1$, i.e., it will be defined on the open interval $(-1, 1)$, but *not* on its endpoints, $a = -1$ or $b = 1$. However, if we have a *polynomial* solution (that is, only finitely many non-zero terms, which happens when $a_{n+2} = 0$ for some n), then this will certainly be defined on all of $[-1, 1]$. What values of λ lead to a polynomial solution? (These are the *eigenvalues* of L.)

(d) The *eigenfunction* for eigenvalue λ_k is a polynomial $P_k(x)$ called the *Legendre polynomial* of degree k. (These arose on HW 2 and HW 5.) By Sturm-Liouville theory, they form an *orthogonal basis* of $\mathsf{L}^2([-1,1])$, meaning that

$$\langle P_n, P_m \rangle := \int_{-1}^{1} P_n(x) P_m(x)\, dx = 0\,, \qquad n \neq m\,.$$

Use the recurrence relation to write out the first five Legendre polynomials, $P_k(x)$, for $k = 0, \dots, 4$. Normalize each one so they form an *orthonormal* set.

(e) Write the polynomial $f(x) = 3x^3 - 2x^2 + 4$ using the first four Legendre polynomials. That is, find C_0, C_1, C_2, and C_3 such that

$$3x^3 - 2x^2 + 4 = C_0 P_0(x) + C_1 P_1(x) + C_2 P_2(x) + C_3 P_3(x)\,.$$

Hint: This is *very* similar to a problem you did on HW 2!